A DETAILED DESCRIPTION OF OGRE, THE OLG MODEL

By Daniel Baksa and Zsuzsa Munkacsi
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Abstract

In this paper we present the structure of OGRE, a dynamic general equilibrium model with overlapping generations, unemployment and a shadow economy. Based on a parametrized version of the model, we examine the impacts of aging and calculate multipliers of public pension and other fiscal policies. Also, we contrast macroeconomic reactions with pay-as-you-go and fully funded pension plans. Lastly, we highlight the role of unemployment and that of the underground sector in the framework.

Keywords: population aging, public old-age pension reforms, pay-as-you-go, fully funded, shadow economy, informal employment, government debt, New Keynesian model, overlapping generations, demography, unemployment, retirement age

JEL classification: E24, E26, H55, J11, J46
1 Novelties and contribution

This paper presents a dynamic general equilibrium model with overlapping generations (OLG) and demography, unemployment and wage bargaining, and an underground sector. It was developed to investigate the macroeconomic effects of a wide range of government policies in what might be termed the “Age of Aging”. The model is called OGRE, an acronym for Overlapping Generations and Retirement.

Classical OLG models, such as Allais (1947), Diamond (1965) and Samuelson (1958), have a finite number of periods; in the simplest version the number of periods is equal to two; agents are young in period 1 and old in period 2. Moreover, agents know when they will die. Taking a different direction, we follow Gertler (1999), who presents a Blanchard-Yaari-type (Blanchard (1985) and Yaari (1965)) OLG model with young and retired agents.\(^1\)\(^2\) Here, agents live forever; however, they die with some probability in each period. We distinguish two cohorts. The first of these is the young (the workers), who\(^3\) either work and pay labor income taxes, or are unemployed and receive unemployment benefits from the government. The second cohort is the old (the retired), who do not work, but receive public old-age pension benefits. Population is not constant over time because young people are born with some probability. Additionally, young people retire with a given probability.

Unemployment is induced by hiring costs; following Blanchard and Gali (2010), workers and firms also bargain over wages. The literature that takes into account unemployment is rather limited; some exceptions include Borsch-Supan et al. (2006), Kilponen et al. (2006), Nickel et al. (2008), Karam et al. (2010), Braz et al. (2013) or McGrattan and Prescott (2015). Furthermore, Corneo and Marquardt (1999) conclude that unemployment is independent of social security systems. In our view, however, it is somewhat misleading to assume full employment. The obvious rigidities in the labor market (job hunting requires money and time on the sides of both the employer and the employee) imply that not everyone seeking employment will find a job, at least not in the short run. Brauninger (2005), Ono (2007), Ono (2010), Marchiori et al. (2011) and de la Croix et al. (2013) claim that there is a relation between unemployment and social security. Like Pierrard and Snessens (2009), Marchiori and Pierrard (2012) and Marchiori and Pierrard (2015), we include unemployment in our framework. To the best of our knowledge, besides Berger et al. (2009)\(^4\), OGRE is the only Blanchard-Yaari-type OLG model with unemployment.

The main novelty of our model is the inclusion of a shadow economy.\(^5\) To the best of our knowledge, OGRE is the first Gertler-type OLG model with informality. We are only aware of one classical OLG model with an informal sector, which is the two-period model of Keuschnigg et al. (2013). In our view, however, our questions cannot

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\(^1\) Unlike Berger et al. (2009), for example, we do not provide a detailed literature review on different types of OLG models.

\(^2\) We use the terms “Blanchard-Yaari-type” and “Gertler-type” interchangeably.

\(^3\) We use the terms “young” and “worker” interchangeably, as we do the terms “old” and “retired”.

\(^4\) Berger et al. (2009) is itself based on Grafenhofer et al. (2006) and Jaag et al. (2007).

\(^5\) We use the terms “shadow”, “informal”, “underground” and “unofficial” interchangeably.
be adequately addressed using two periods, as it is crucial to examine both short- and long-run responses.

While on the one hand, the underground sector is modeled by tax evasion, on the other, the level of regulation is lower in the shadow; here we follow Williamson (1975). Taxes are avoided in both the informal labor and goods markets. Specifically, i) labor income is only taxed in the formal sector (both that of employers and employees) and ii) value-added taxes are only paid for goods purchased from formal sector-producers. Then, the fact that the formal sector is more regulated than the informal one is modeled by higher labor and product market rigidities in the former. Furthermore, the government only buys formal goods.

We contend that, for several reasons, neglecting the shadow economy leaves a framework markedly incomplete. First, as the shadow economy is by definition in the shadow, public policies can only affect directly the non-shadow side of the economy. Consequently, the larger the underground economy, the smaller the fraction of the economy directly influenced by the government. The reason for this is that only formal-sector firms and workers pay taxes, and only they are directly affected by the level of regulation (unionization e.g.).

Second, the official and unofficial sectors interact. For instance, workers move between them. Moreover, the higher the size of informality, the more workers can potentially move out of the shadow. The same is true for goods. Furthermore, some studies note that working in the shadow has been theorized to function as a kind of “insurance policy”, in other words, the “shadow employment is tolerated because its repression increases unemployment” (Boeri and Garibaldi (2007), page 125).

Third, working in the shadow is relevant not only because it affects public revenues in general, but also because it is related to social security, in particular. In a pay-as-you-go (PAYG or PG) regime, revenues collected today are used to finance public old-age pension spending today; those who work informally do not contribute to social security; hence shadow work reduces pension benefits or increases public debt. At the same time, in a fully funded (FF) system the current social security contributions deducted from firms and workers finance future public old-age pension benefits of the same people. Thus, working in the shadow means no pension savings for the period spent in the shadow. As a consequence, although in the short run underground employment might function as “insurance”, in the long run it might be costly for both the people and the state.

Last, in most countries the shadow output is rather sizable (Schneider et al. (2010)). This is true for many developed economies as well as for emerging economies.  

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6Also, as Schneider (2012) (page 6) claims: “The shadow economy includes all market-based legal production of goods and services that are deliberately concealed from public authorities for the following reasons: 1. to avoid payment of income, value added or other taxes, 2. to avoid payment of social security contributions, 3. to avoid having to meet certain legal labor market standards, such as minimum wages, maximum working hours, safety standards, etc., and 4. to avoid complying with certain administrative obligations, such as completing statistical questionnaires or other administrative forms.”

7Informality can also be measured in terms of employment (Schneider (2012)).
OGRE is suitable for investigating the impacts of retirement age changes, thus further contributing to the literature. To the best of our knowledge, with the exception of Fehr (2000), Kilponen et al. (2006), Diaz-Gimenez and Diaz-Saavedra (2009), Keuschnigg et al. (2013) and Goraus et al. (2014), only social security contribution rates and pension-wage replacement rates have been considered in the research literature. Additionally, to date, the macroeconomic reactions of pension reforms have not been compared with those of other public policies.

OGRE’s final contribution is its rich fiscal sector. Public revenues are labor income taxes (personal income tax, employee and employer social security contributions) and value-added tax, while the government finances government consumption, unemployment benefits and old-age pensions. Two pension plans are distinguished. In the first of these pension benefits are based on previous wage stream and a pension-wage replacement rate. According to World Bank (1994) there are two main subtypes of PAYG plans: a “defined benefit scheme,” which fixes pensions and allocates taxes accordingly, and a “defined contribution scheme,” which fixes taxes that are then redistributed. Our model is closer to the first of these. In practice, however, most systems are a combination of the two plans. By fully funded we mean a system where each worker’s contribution is collected in a separate account and are used to finance the future pension benefits of the account owner himself/herself.

Like Borsch-Supan and Ludwig (2011), we study both PAYG and FF pension plans. Most models in the literature either incorporate a PAYG plan (e.g. Nickel et al. (2008) and Karam et al. (2010), but also the AINO, PESSOA and LOLA models), or investigate a switch towards an FF system (e.g. Borsch-Supan et al. (2006) and McGrattan and Prescott (2015)), but very few include both (exceptions are Marchiori et al. (2011) and de la Croix et al. (2013)).

The model’s main channels of aging are as follows. If either the mortality or the fertility rate goes down, there are relatively fewer young and more old people. Hence, the size of the economy shrinks, which implies that employment and thus demand are negatively affected. Also, public debt goes up. The short- and long-run implications of different demographic shocks differ somewhat, as do those of public policies, which might prove pivotal in managing the consequences of aging. Importantly, labor income tax cuts imply a rise in the unemployment rate while other fiscal expansion policies do not distort the labor market in the long run. Furthermore, there are short-run costs; for instance, a temporary rise in unemployment occurs after a cut in VAT or an increase in government consumption.

Workers’ incentives differ between PAYG and FF pension plans. Hence, the paths of public debt also differ in the two regimes. Nonetheless, we find that the long-run macroeconomic responses of fiscal policies do not significantly differ. In the short run,
however, the composition of private consumption reacts differently; namely, in a fully funded pension regime, a shift towards young generations usually takes place. As regards aging, the composition of consumption reacts differently with different pension regimes in the long run as well.

The presence of unemployment and informality is crucial for long-run responses of household consumption and GDP of labor income tax policies; without them the multipliers are much lower. When the shadow economy is not taken into account, as is true in the other models, the tax base is larger, allowing an artificially low tax reduction to induce the same size shock. In the short term, unemployment is also impacted and so is the consumption basket. The intuition is that without a shadow economy those who leave formal employment are considered to be unemployed instead of moving into the shadow. Even the signs of some short-run multipliers change. Thus, a model without informality and/or unemployment might bias the short-term reactions, which would result in incorrect policy conclusions.

In the next section we describe the model in detail. Then, based on a parametrized economy, we present the impacts of aging and calculate fiscal multipliers. Also, we compare macroeconomic reactions in the two pension regimes and show evidence of the role of informality and unemployment. The Appendix provides information on old and young generations’ optimization and details derivations related to the two pension systems. Finally, we list all the normalized equations and show the steps we took in calculating the model’s steady state. Throughout the paper we emphasize long-run versus short-run differences.

2 The model

OGRE is a dynamic general equilibrium model with overlapping generations (OLG), unemployment and an underground sector. OGRE is an acronym for Overlapping Generations and Retirement. OGRE’s main novelty is the distinction it makes between formal and informal labor and goods markets in an OLG setup.

2.1 Demography

Demography is modeled by Gertler (1999), Blanchard (1985) and Yaari (1965). Two cohorts are distinguished: the young (the workers) and the old (the retired). Population is not constant over time, as young people are born and old people pass away with some probability. Also, young people retire with a given probability.

The total number of population is \( N_t \) which is the sum of young/worker \((N^Y_t)\) and old/retired \((N^O_t)\) people. Young people retire with a probability of \( \omega_{t-1}^Y \), while old people die with a probability of \( \omega_{t-1}^O \). Furthermore, \( n_t \) is the fertility rate which shows the birth rate of new young (worker) people. This is a net rate; we do not model those who do not work due to their age (students), the mortality of the young, or migration. So, the relevant demographic equations are:

\[
N_t = N^Y_t + N^O_t
\]
\[ N_t^Y = (1 - \omega_{t-1}^Y)N_{t-1}^Y + n_tN_{t-1}^Y \]
\[ N_t^O = (1 - \omega_{t-1}^O)N_{t-1}^O + \omega_{t-1}^YN_{t-1}^Y \]

### 2.2 Overlapping Generations

Overlapping generations are also modeled following Gertler (1999), Blanchard (1985) and Yaari (1965), in a system called the Blanchard-Yaari framework. The young (the workers) either work and pay labor income taxes, or are unemployed and receive unemployment benefits from the government. Following Blanchard and Gali (2010), unemployment is induced by hiring costs; workers and firms also bargain over wages. The old (the retired) do not work, but receive public old-age pension benefits from the government. We describe the retired cohort first, and the young cohort afterwards.

#### 2.2.1 The retired cohort

Retired agent \( i \) of retired cohort \( a \) is one individual who retired \( a \) years ago. He or she maximises the following Bellman equation:

\[
V^{O}(B_{a-1,t-1}(i)) = \max \left\{ \left(1 + \epsilon_t^C \right) \left[ \frac{1}{1-\gamma} \left\{ C_{a,t}^{OF}(i) \right\}^{1-\gamma} + \frac{\chi}{1-\gamma} \left\{ C_{a,t}^{OI}(i) \right\}^{1-\gamma} \right] + \beta E_t(1 - \omega_t^O)V^{O}(B_{a,t}(i)) \right\}
\]

subject to this budget constraint:

\[
(1 + \tau_t^C)C_{a,t}^{OF}(i) + p_t^I C_{a,t}^{OI}(i) + (1 - \omega_t^O)B_{a,t}(i) = (1 + r_{t-1})B_{a-1,t-1}(i) + TR^{PG,YO}_{a,t}(i) + TR^{FF,YO}_{a,t}(i) + \text{Profit}_{a,t}(i) - T_{O,a,t}(i)
\]

The retired agent does not work, but receives pension benefits from the government: \( TR^{PG,YO}_{a,t}(i) \) in a PAYG regime and \( TR^{FF,YO}_{a,t}(i) \) in a fully funded regime (where \( YO \) denotes pension benefits of those who just retired (i.e., those who were young one period before)). Pension benefits of the newly retired are determined in the period in which they retire and are indexed by inflation.\(^{11} \) Also, some share of profits minus lump-sum taxes are received by the retired. The agent consumes formal and informal goods. Specifically, its utility depends on consuming goods produced by formal firms \( (C_{a,t}^{OF}(i)) \) and consuming goods produced by informal firms \( (C_{a,t}^{OI}(i)) \). \( \chi \) parameter shows that formal and informal goods are differently valued. Namely, formal and informal goods are not perfect substitutes, because there is no warranty for informal goods. So, one unit of an informal good implies a lower utility than one unit of a formal good. The agent pays value-added taxes (VAT) after purchasing formal goods; VAT is denoted by \( \tau_t^C \). Because informal goods are hidden, no VAT is paid upon their purchase. \( p_t^I \) is the relative price of informal goods (expressed in formal goods’ price level), \( \epsilon_t^C \) is the preference (demand) shock and \( \gamma \) is the relative risk aversion parameter. Finally, besides consuming, the agent saves in \( B_{a,t}(i) \) risk-free bonds and receives \( r_{t-1} \) real interest rate (nominal interest rate is denoted by \( i_{t-1} \)) on the previous period’s bond holdings. The retired agent optimizes with respect to \( C_{a,t}^{OF}(i) \), \( C_{a,t}^{OI}(i) \) and \( B_{a,t}(i) \), and, when optimizing, he or she also takes into account that with probability \( \omega_t^O \) by the beginning of the next period he or she will pass away.

\(^{11}\) More detail is available in Section 2.5.
As a result, the Euler-equation for formal goods is:

\[ E_t C_{a+1,t+1}^{O,F}(i) = E_t C_{a,t}^{O,F}(i)(1 + r_t)^{1/\gamma} \Lambda_{t+1} \]

where

\[ E_t \Lambda_{t+1} = E_t \left\{ \beta \frac{1 + \epsilon_{t+1}^C}{1 + \epsilon_t^C} \frac{1 + \tau_{t+1}^C}{1 + \tau_t^C} \right\}^{\frac{1}{\gamma}} \]

The Euler-equation shows the usual intertemporal substitution between periods \( t \) and \( t + 1 \); it is also affected by value-added tax rates.

Then, optimization also implies that informal consumption can be expressed in terms of formal consumption as follows:

\[ C_{a,t}^{O,I}(i) = \Upsilon_t C_{a,t}^{O,F}(i) \]

where

\[ \Upsilon_t = \left\{ \chi \frac{1 + \tau_t^C}{p_t^I} \right\}^{\frac{1}{\gamma}} \]

The higher the parameter \( \chi \) and the level of VAT, the larger the relative value of informal consumption to formal consumption. The opposite is true regarding the relative price of informal goods.

As noted above, the retired agent knows that he or she will die with a probability \( \omega_t^O \) by the beginning of the next period. As a consequence, the agent changes his or her behavior because his or her optimization horizon becomes finite. In contrast to a representative household framework, where, in the long run, the stochastic discount factor is equal to the inverse of one plus the real interest rate, in an overlapping generation setup this is not the case. Rather, the stochastic discount factor is a weighted sum of adjacent periods’ real interest rates, where the weights are related to the demographic environment, namely, to the probabilities of death (and the retirement probability for the young).

For instance, in period \( t \) the probability of death today is \( \omega_t^O \); thus, the survival probability is \( 1 - \omega_t^O \). Then, the same agent’s \( t+1 \)-period survival rate today is \( (1 - \omega_t^O)(1 - \omega_{t+1}^O) \).

As a consequence, the future stream of income must be discounted with the survival probabilities, and not only with the usual real interest rates, because the agent might die in any period. The same is true for retirement. Both death and retirement change the current state of the agent, and so they must be taken into account during the optimization process. As such, discount rates, and consequently, aggregation, are affected. In a representative agent framework there is always only one state, thus making discounting, and thus aggregation, simple. In an overlapping generations setup, however, both discounting and aggregation are affected by demography.

We can show that the current-period individual formal consumption level of retired agent \( i \) can be expressed as follows:

\[ \mathcal{H}_t^O C_{a,t}^{O,F}(i) = (TR_{a,t}^{P,G,YO}(i) + TR_{a,t}^{F,F,YO}(i))\Omega_t^O + I_{a,t}(i) + (1 + r_t - 1)B_{a-1,t-1}(i) \]
where

\[
T_{a,t}^O(i) = \text{Profit}_{a,t}^O(i) - T_{a,t}^O(i) + E_t \frac{1 - \omega_t^O}{1 + r_t} T_{a+1,t+1}^O(i)
\]

\[
H_t^O = (1 + \tau_t^C) + p_t^I \Upsilon_t + E_t (1 - \omega_t^O) (1 + r_t) \frac{1}{\Lambda_{t+1}} \Omega_{t+1}^O
\]

\[
\Omega_t^O = 1 + E_t \frac{1 - \omega_t^O}{1 + r_t} \Omega_{t+1}^O
\]

and \( \Upsilon_t \) and \( \Lambda_{t+1} \) are the same as before.

Then, after aggregation, the retired generation consumes:\(^{12}\)

\[
H_t^O C_t^{O,F} = (TR_t^P + TR_t^FF) \Omega_t^O + T_t^O + (1 + r_{t-1})(\omega_t^{Y_{t-1}} B_{t-1}^Y + B_{t-1}^O)
\]

\[
C_t^{O,I} = \Upsilon_t C_t^{O,F}
\]

where

\[
T_t^O = \text{Profit}_t^O - T_t^O + E_t \frac{1 - \omega_t^O}{(1 + r_t)(1 + g_{t+1}^{NO})} T_{t+1}^O
\]

\[
\text{Profit}_t^O - T_t^O = (1 - \xi)(\text{Profit}_t - T_t)
\]

and \( \xi \) is the fraction of profits and lump-sum taxes that goes to the young.

Hence, the current level of formal consumption of the old equals the sum of the discounted stream of current and future pension benefits and other income, and the current level of savings. Discounting not only depends on the usual real interest rates, however, but also on mortality rates and the number of people in the retired cohort. A final crucial caveat is that \( \omega_t^{Y_{t-1}} B_{t-1}^Y \) is included in the consumption function because \( \omega_t^{Y_{t-1}} \) share of the young retired in the previous period. Also, \( g_{t+1}^{NO} \) shows the growth rate of the number of old people.

### 2.2.2 The young cohort

Young agent \( i \) of young cohort \( b \) is one individual of its cohort who started to work (i.e., was born) \( b \) years ago. The Bellman-equation of a young individual is:

\[
V_t^Y(B_{b-1,t-1}(i)) = \max \left\{ (1 + \epsilon_t^G) \left[ \frac{1}{1 - \gamma} \left\{ C_{b,t}^{Y,F}(i) \right\}^{1-\gamma} + \frac{X}{1 - \gamma} \left\{ C_{b,t}^{Y,I}(i) \right\}^{1-\gamma} \right] + \right.
\]

\[
\left. + \beta E_t \left( (1 - \omega_t^Y) V_{t+1}^Y(B_{b,t}(i)) + \omega_t^Y V_{t+1}^O(B_{b,t}^O(i)) \right) \right\}
\]

while the budget constraint is:

\[
(1 + \tau_t^C) C_{b,t}^{Y,F}(i) + p_t^I C_{b,t}^{Y,I}(i) + (1 - \omega_t^Y) B_{b,t}(i) + \omega_t^Y B_{b,t}^O(i) =
\]

\[
= (1 + r_{t-1}) B_{b-1,t-1}(i) + (1 - \tau_t^{LW}) w_t^F L_t^F(i) + w_t^I L_t^I(i) +
\]

\[
+ w_t^U U_{b,t}(i) + \text{Profit}_{b,t}^Y(i) - T_{b,t}^Y(i)
\]

\(^{12}\)The aggregate informal consumption of the retired can be calculated by the informal-formal substitution equation.
By definition, a young person does not collect old-age pension benefits. Rather, he or she either works and receives labor income, or is unemployed ($U_{b,t}(i)$ denotes unemployment) and receives $w_t^U$ unemployment benefits. The agent can either work in the formal sector or in the informal sector; $L_{b,t}^F(i)$ and $L_{b,t}^I(i)$ denote formal and informal employment, and $w_t^F$ and $w_t^I$ are formal and informal wages. A crucial difference between the formal and informal sectors is that only income earned in the formal sector is subject to taxation; those who work in the formal sector pay a sum of personal income tax and employees’ social security contributions of $\tau_t^{LW}$. Also, a given share of Profit$_{b,t}(i)$ profits minus $T_b^Y(i)$ lump-sum taxes are earned by the young, where the share is equal to the fraction of young people in the whole population. The probability of retiring by the next period is $\omega_t^Y$. A young agent saves for two possible future states: first, for staying young ($B_{b,t}(i)$); and second, for retiring ($B_{b,t}^{YO}(i)$). The young optimize with respect to both savings, i.e. $B_{b,t}^Y(i)$ and $B_{b,t}^{YO}(i)$, and, as usual, $C_{b,t}^{Y,F}(i)$ and $C_{b,t}^{Y,I}(i)$.

As a result, the Euler-equations are:

$$E_t C_{b,t+1,t+1}^{Y,F}(i) = E_t C_{b,t}^{Y,F}(i)(1 + r_t)\frac{1}{1 + r_t} \Lambda_{t+1}$$

$$E_t C_{b,t+1,t+1}^{O,F}(i) = E_t C_{b,t}^{Y,F}(i)(1 + r_t)\frac{1}{1 + r_t} \Lambda_{t+1}$$

where $\Lambda_{t+1}$ is the same as before. The young agent saves for two potential future states; hence, there are two Euler-equations. These show the intertemporal substitution, as usual. However, one of them relates current-period young consumption to next-period young consumption, if the agent is still young in the next period, while the other one does the same but in relation to future old consumption.

Also, the relation between informal and formal consumptions is:

$$C_{b,t}^{Y,I}(i) = \Upsilon_t C_{b,t}^{Y,F}(i)$$

where $\Upsilon_t$ is the same as before. As with regard to the old, informal consumption is positively affected by the weight on informal consumption in the utility function and the rate of value-added tax, but it is negatively affected by the relative price of informal to formal goods.

Then, individual formal consumption of the young are:

$$H_t^Y C_{b,t}^{Y,F}(i) = \mathcal{I}_{b,t}(i) + \frac{\mathcal{I}_{b,t}^{YO}(i)}{1 + r_t} + (1 + r_{t-1})B_{b-1,t-1}(i)$$

where

$$\mathcal{I}_{b,t}(i) = Inc_{b,t}(i) + E_t \frac{1 - \omega_t^Y}{1 + r_t} \mathcal{I}_{b,t+1}(i)$$

$$Inc_{b,t}(i) = (1 - \tau_t^{LW}) w_t^F L_{b,t}^F(i) + w_t^I L_{b,t}^I(i) + \omega_t^Y U_{b,t}(i) + Profit_{b,t}(i) - T_b^Y(i)$$

$$\mathcal{I}_{b,t}^{YO}(i) = E_t \omega_t^Y \left( (TR_{0,t+1}^{PG,YO}(i) + TR_{0,t+1}^{FYO}(i)) \Omega_{t+1} + \mathcal{I}_{0,t+1}(i) \right) +$$

$$+ E_t \frac{1 - \omega_t^Y}{1 + r_{t+1}} \mathcal{I}_{b,t+1}(i)$$

$$H_t^Y = (1 + \tau_t^C) + p_t \Upsilon_t + E_t (1 + r_t) \frac{1}{1 + r_t} \Lambda_{t+1} \left( (1 - \omega_t^Y) H_{t+1}^Y + \omega_t^Y H_{t+1}^O \right)$$

13This is just a technical distinction; at the end of the day, all young persons’ savings are denoted by $B_{b}$.
where $\Upsilon_t$ and $\Lambda_{t+1}$ are the same as before.

Aggregate consumption is as follows:

$$\mathcal{H}_t^Y C_t^{Y,F} = \mathcal{I}_t^Y + \frac{\mathcal{I}_{t+1}^{Y,I}}{1 + r_t} + (1 + r_{t-1})(1 - \omega^Y_{t-1})B_{t-1}^Y$$

$$C_t^{Y,I} = \Upsilon_t C_t^{Y,F}$$

where

$$\mathcal{I}_t^Y = Inc_t + p_t^{L}L_t^F + w_t^{L}L_t^I + w_t^{U}U_t + Profit_t^Y - T_t^Y$$

$$Profit_t^Y - T_t^Y = \xi(Profit_t - T_t)$$

$$\mathcal{I}_{t+1}^{I,Y} = E_t \left( (TR_{t+1}^{PG,Y} + TR_{t+1}^{FF,Y})\Omega_{t+1} + \frac{\omega^Y_t}{(1 + g_{t+1}) s_{t+1}} \mathcal{I}_{t+1}^O \right) +$$

$$+ E_t \frac{1 - \omega^Y_t}{(1 + r_{t+1}) (1 + \omega^N_{t+1})} \mathcal{I}_{t+1}^O$$

and $\Upsilon_t$, $\Lambda_{t+1}$ and $\xi$ are the same as before.

Like retired consumption, the current level of young formal consumption is equal to the sum of the discounted stream of current and future income, and current savings. Nevertheless, there are two future incomes, one for being young in the future and one for being old in the future. Moreover, income denotes labor income and other income (but not pensions). Finally, discounting takes into account not only mortality rates and the number of retired people, but also the probability of retirement and the number of young people.

Finally, the aggregate young budget constraint is:

$$(1 + \tau_t^C)C_t^{Y,F} + p_t^I C_t^{Y,I} + B_t^Y = Inc_t + (1 + r_{t-1})(1 - \omega^Y_{t-1})B_{t-1}^Y$$

### 2.3 Firms

There are two types of firms: physical-capital producing firms and goods-producing firms; both types are owned by the household sector, and, among both types there are formal and informal firms. Informal firms avoid paying taxes and, as they are in the shadow sector, the level of regulation is lower (Williamson (1975)).

Physical-capital producing firms use beginning-of-period physical capital and invest to produce end-of-period physical capital. Following Christiano et al. (2005), investment is subject to an investment-adjustment cost.
Goods-producing firms rent physical capital from capital producers and hire labor from young households. They produce a good which is either consumed by households or purchased by the government (only formal sector goods), or one that is used for capital production (only formal sector goods). Also, only formal firms pay social security contributions. All goods-producing firms pay an extra cost of hiring following Blanchard and Gali (2010); this hiring cost depends on the number of newly hired people. The hiring cost is higher in the formal sector, reflecting both administration and advertising costs. While hiring is endogenous, firing is exogenous. Furthermore, goods producers are monopolistically competitive and set the price level taking into account a price adjustment cost à la Rotemberg (1982). Finally, goods-producing firms bargain with workers over wages, similarly to Blanchard and Gali (2010). Formal workers have a stronger bargaining power over wages than do informal workers, and, thanks to unions, the firing probability is lower in the formal sector.

2.3.1 Physical capital producers

The Bellman equation of the formal physical capital producer is:\(^{14}\)

\[
V(\text{Inv}_{t-1}^F; K_{t-1}^F) = \max \left\{ r_t^{K,F} K_{t-1}^F - \text{Inv}_t^F + E_t \frac{1}{1 + r_t} V(\text{Inv}_{t-1}^F; K_t^F) \right\}
\]

while it also takes into account the physical capital accumulation equation:

\[
\text{Inv}_t^F = K_t^F - (1 - \delta) K_{t-1}^F - \text{Inv}_t^F S \left( \frac{\text{Inv}_t^F}{\text{Inv}_{t-1}^F (1 + g_t)} \right)
\]

Here, \( r_t^{K,F} \) denotes the real rental rate of capital (\( R_t^{K,F} \) is the nominal rental rate of capital and \( r_t^{K,F} = \frac{R_t^{K,F}}{p_t} \)), \( K_{t-1}^F \) is the physical capital stock, \( \text{Inv}_t^F \) denotes investment, \( Q_t^F \) is the shadow price of capital and \( \delta \) is the depreciation rate of capital. Also, \( g_t \) is the sum of technology and population growth.

Function \( S \) is the investment adjustment cost function; following Christiano et al. (2005), it looks like:

\[
S \left( \frac{\text{Inv}_t^F}{\text{Inv}_{t-1}^F (1 + g_t)} \right) = \frac{\phi_{Inv}^F}{2} \left( \frac{\text{Inv}_t^F}{\text{Inv}_{t-1}^F (1 + g_t)} \right)^2
\]

where \( \phi_{Inv}^F \) is the investment adjustment cost parameter.

As a result, optimization can be summarized by two equations:

\[
E_t \left( r_{t+1}^{K,F} + Q_{t+1}^F (1 - \delta) \right) = Q_t^F (1 + r_t)
\]

\[
1 = E_t \frac{1}{1 + r_t} \left[ Q_{t+1}^F S' \left( \frac{\text{Inv}_{t-1}^F}{\text{Inv}_t^F (1 + g_{t+1})} \right) \left( \frac{\text{Inv}_{t+1}^F}{\text{Inv}_t^F (1 + g_{t+1})} \right)^2 + \right. + Q_t^F \left. \left[ 1 - S \left( \frac{\text{Inv}_t^F}{\text{Inv}_{t-1}^F (1 + g_t)} \right) - S' \left( \frac{\text{Inv}_t^F}{\text{Inv}_{t-1}^F (1 + g_t)} \right) \frac{\text{Inv}_t^F}{\text{Inv}_{t-1}^F (1 + g_t)} \right] \right]
\]

\(^{14}\)The optimization problem of the informal physical capital producer is parallel, so we only describe the formal one.
The first equation is the usual no-arbitrage condition. It claims that, taking into account the shadow price of capital $Q_t^F$, investing in capital and saving in risk-free bonds yields the same return. Without this condition a sub-optimal arbitrage could be achieved in the economy. The second equation is the usual Tobin-Q, which describes the optimal investment strategy of the firm.

### 2.3.2 Good producers

We now turn to goods producers in the formal sector. One of them is denoted by $j$. The (nominal) Bellman-equation of the formal sector goods producer is:

$$V(P_{t-1}^F(j), L_{t-1}^F(j)) = \max \left\{ \text{profit}_t^F(j) + E_t \frac{V(P_t^F(j), L_t^F(j))}{1 + \pi_t} \right\}$$

which is maximized subject to the profit function, production function, demand function and the labor law of motion with respect to $P_t^F(j), L_t^F(j), K_t^F(j)$ and $H_t^F(j)$:

$$\text{profit}_t^F(j) = P_t^F(j)Y_t^F(j) - R_t^K K_t^F(j) - \left(1 + \tau_t^{SSCF}\right)W_t^F L_t^F(j) - HC_t^F H_t^F(j) - P_t^F(j)Y_t^F R \left(\frac{P_t^F(j)}{P_{t-1}^F(j)}\right)$$

$$Y_t^F(j) = K_{t-1}^F(j)^{\alpha_F} (A_t^F L_t^F(j))^{1-\alpha_F}$$

$$Y_t^F(j) = \left(\frac{P_t^F(j)}{P_{t-1}^F(j)}\right)^{-\varphi} Y_t^F$$

$$L_t^F(j) = (1 - pr^{F,F})L_{t-1}^F(j) + H_t^F(j)$$

$$R \left(\frac{P_t(j)}{P_{t-1}(j)}\right)^{-\gamma} = \phi_P \left(\frac{P_t^F(j)}{P_{t-1}^F(j)} - 1\right)^2 = \phi_P \frac{1}{2} \left(\frac{1 + \pi_t^F}{(1 + \pi_{t-1})^\gamma} - 1\right)^2$$

$$HC_t^F = \kappa_F \left(pr_t^{H,F}\right)^{a_{HC}}$$

$$pr_t^{H,F} = \frac{H_t^F}{U_{t-1} + pr^{F,F} L_{t-1} + pr^{F,I} L_{t-1}}$$

First, regarding production, the production function is the usual Cobb-Douglas with a labor-augmenting technological process based on labor and physical capital. Here, $A_t^F$ denotes technology level and $1 - \alpha_F$ shows the share of labor income in total factor income. The total amount of production $Y_t^F$ is related to firm $j$-production $Y_t^F(j)$ by the usual demand function, where $P_t^F(j)$ is the price set by firm $j$ ($P_t^F$ is the general price level) and $\varphi$ denotes price elasticity of demand.

Second, the extra cost of hiring a new worker is denoted by $HC_t^F$ (the real hiring cost is $hc_t^F = HC_t^F / p_t^F$). This depends on the probability of hiring $pr_t^{H,F}$, which in turn depends on the number of newly hired people $H_t^F$. $\kappa_F$ denotes the hiring cost parameter and $a_{HC}$ shows the elasticity of hiring cost with respect to the probability of hiring. While hiring is endogenous, as the labor law of motion shows, firing is exogenous, and the probability of firing is $pr^{F,F}$. Also, firms pay $\tau_t^{SSCF}$ social security contributions.

---

15 The optimization problem of informal goods producers is parallel.

16 Because the number of firms is equal to 1, index $j$ is only used for the sake of mathematical convenience; at the end, a variable with $j$ and without $j$ will be the same.
Finally, regarding price setting, function $R$ is the Rotemberg price adjustment cost. Also, the price adjustment cost parameter is $\phi_F^P$ and $\gamma$ is the indexation parameter; if $\gamma = 0$, there is no indexation, if $\gamma = 1$, there is full indexation.

Although sectoral optimizations are similar, some differences ought to be noted. First and foremost is tax evasion; that is, only formal firms pay social security contributions. Moreover, formal hiring costs and bargaining power of workers are higher, while the dismissal rate is lower. These differences can be attributed to the higher degree of regulation in the formal as compared to the informal sector (Williamson (1975)).

Optimization results in two demand functions (physical capital and labor) and a pricing decision, while the last equation is the real marginal cost ($MC_t^F = \frac{MC^F}{p_t}$):

\[
K_{t-1}^F = \alpha^F \frac{mc_t^F}{r_t} V_t^F
\]

\[
mc_t^F (1 - \alpha^F) Y_t^F \left( 1 + \frac{1}{\varphi - 1} \left\{ R \left( \frac{1 + \varphi^F_1}{(1 + \varphi^F_{t-1})^\gamma} \right) + R' \left( \frac{1 + \varphi^F_1}{(1 + \varphi^F_{t-1})^\gamma} \right) \left( 1 + \frac{1}{\varphi - 1} \right) \right\} - E_t \frac{1}{\varphi - 1} \right) \]

\[
\frac{1 + \varphi^F_1}{(1 + \varphi^F_{t-1})^\gamma} \frac{1}{1 + \tau_t} Y_{t+1}^F R' \left( \frac{1 + \varphi^F_1}{(1 + \varphi^F_{t-1})^\gamma} \right) \left( 1 + \frac{1}{\varphi - 1} \right) = \varphi - 1 \frac{mc_t^F}{\varphi - 1} \]

\[
mc_t^F = E_t \left( \frac{K_t^F}{\alpha^F} \right)^{\alpha^F} \left( 1 + \frac{1}{\varphi - 1} \right) \frac{1}{1 + \tau_t} Y_{t+1}^F R' \left( \frac{1 + \varphi^F_1}{(1 + \varphi^F_{t-1})^\gamma} \right) \left( 1 + \frac{1}{\varphi - 1} \right) \]

Two important caveats follow. First, due to the presence of labor market frictions, labor demand becomes an intertemporal as well as an intratemporal decision. This is so because labor market rigidities create a link in employment today and tomorrow, a link that is described by the labor law of motion. Second, the pricing decision is sectoral; hence, there are two New Keynesian Philips curves, one for each sector.

Goods producers produce goods and set prices, but they also bargain over wages with their workers. Because an agent works in the formal sector, in the informal sector, or is unemployed, there are three worker value functions:

\[
V_t^F = (1 - \tau_t^{\text{lw}}) w_t^F + E_t \frac{1}{1 + \tau_t} \left[ (1 - pr_t^{F,F} + pr_t^{F,F} pr_{t+1}^{H,F}) V_{t+1}^F \right. + \left. pr_t^{F,F} pr_{t+1}^{H,I} V_{t+1}^I + pr_t^{F,F} (1 - pr_t^{H,F} - pr_{t+1}^{H,F}) V_{t+1}^U \right]
\]

\[
V_t^I = w_t^I + E_t \frac{1}{1 + \tau_t} \left[ (1 - pr_t^{F,I} + pr_t^{F,I} pr_{t+1}^{H,I}) V_{t+1}^I \right. + \left. pr_t^{F,I} pr_{t+1}^{H,F} V_{t+1}^F + pr_t^{F,I} (1 - pr_t^{H,F} - pr_{t+1}^{H,F}) V_{t+1}^U \right]
\]

\[
V_t^U = w_t^U + E_t \frac{1}{1 + \tau_t} \left[ (1 - pr_t^{H,I} - pr_t^{H,F}) V_{t+1}^U + pr_t^{H,F} V_{t+1}^I + pr_t^{H,I} V_{t+1}^I \right]
\]
Today, a worker in the formal sector earns real wage $w^F_t$ and pays personal income tax and social security contributions $\tau_{LW}^t$. In the next period, she might keep this job or might be fired. If fired, she could find another job either in the formal or in the informal sector. Alternatively she might stay unemployed. A similar argument holds for the informal worker value function. Being unemployed, nevertheless, means that $w^U_t$ unemployment benefits are received today, and tomorrow she either starts to work in any of the two sectors or stays unemployed for one more period.

Firm value functions are equal to the hiring costs themselves because searching does not take time (see Blanchard and Gali (2010) for more detail).

Bargaining happens separately in each sector with bargaining powers $\sigma^F$ and $\sigma^I$; workers and firms make a common decision about wages, taking into account the value of the job over that of the outside option (i.e., not working): 

$$\max_{w^F_t} (V^F_t - V^U_t)^{\sigma^F} (hc^F_t)^{1-\sigma^F}$$

$$\max_{w^I_t} (V^I_t - V^U_t)^{\sigma^I} (hc^I_t)^{1-\sigma^I}$$

As a result, the formal bargaining condition is:

$$\frac{\sigma^F}{1-\sigma^F} hc^F_t \frac{1 - \tau^LW_t}{1 + \tau^SSCF_t} = (1 - \tau^LW_t) w^F_t - w^U_t +$$

$$+ E_t \frac{1}{1 + r_t} \left[ (1 - pr^F,F_t)(1 - pr^H,F_{t+1}) \left( \frac{\sigma^F}{1 - \sigma^F} hc^F_{t+1} \frac{1 - \tau^LW_{t+1}}{1 + \tau^SSCF_{t+1}} \right) -
$$

$$- (1 - pr^F,F_t) pr^H,I_{t+1} \left( \frac{\sigma^I}{1 - \sigma^I} hc^I_{t+1} \right) \right]$$

Note, that both employer and employee taxes affect bargaining. Also, bargaining, like labor demand, is an intertemporal decision, due to the presence of hiring costs.\textsuperscript{17}

\section{2.4 Monetary policy}

In the short run, price rigidity matters and the central bank follows a Taylor-type rule, similar to that of Smets and Wouters (2007):

$$1 + i_t = (1 + i_{t-1})^{\rho_i} E_t \left( (1 + r_t)(1 + \pi^F_{t+1})^{\rho_\pi} \right)^{(1-\rho_i)} e^{i_t}$$

Here, $\rho_i$ is the interest rate smoothing parameter, $\rho_\pi$ is the weight on inflation and $e^{i_t}$ is the monetary policy shock. We assume, for the sake of simplicity, that the central bank sets the interest rate based solely on the formal inflation rate.

Also, the Fisher equation is:

$$1 + i_t = (1 + r_t) E_t(1 + \pi^F_{t+1})$$

\textsuperscript{17}There is a similar expression for the informal sector.
2.5 Fiscal policy

The fiscal side of our model is very rich, especially with respect to pensions. First, we present the pay-as-you-go-plan, then we move on to the fully funded regime. We further highlight how we model the switch between these two.

2.5.1 Pay-as-you-go pension system

The government collects revenues ($Rev_t$) of value-added taxes ($\tau^C_t$), labor income taxes ($\tau^L_t$) and $T_t$ lump-sum taxes:

$$Rev_t = \tau^C_t C^F_t + \tau^L_t L^F_t + T_t$$

$$\tau^L_t = \tau^{PI}_t + (1 - \Xi)(\tau^{SSCW}_t + \tau^{SCF}_t)$$

$$\tau^{LW}_t = \tau^{PI}_t + \tau^{SSCW}_t$$

Labor income taxes are paid by young households and firms; the former pay personal income taxes ($\tau^{PI}_t$) and social security contributions ($\tau^{SSCW}_t$) ($\tau^{LW}_t$ denotes the sum of these two), while firms are only subject to social security contributions ($\tau^{SCF}_t$). $\Xi$ is an indicator, which is 1 in a fully funded regime and 0 in a PAYG regime, while it can be time-variant as well. Obviously, value-added taxes are only collected after goods purchased from formal sector producers, and, only firms and workers in the formal sector pay labor income taxes.

Revenues finance government consumption expenditure ($Gov_t$), unemployment benefits expenditure and old-age pension expenditure ($TR^{PG}_t$):

$$Exp_t = Gov_t + w_U^t U_t + TR^{PG}_t$$

As the budget might not be balanced in each period, the government issues bonds denoted by $B_t$. These can be purchased by all households. Thus, the government budget constraint is:

$$B_t + Rev_t = (1 + r_{t-1})B_{t-1} + Exp_t$$

Also, primary government balance ($PB_t$) is defined as the difference between public revenues and expenditures, while total government balance ($GB_t$) also incorporates interest payments/receipts on bond holdings:

$$PB_t = Rev_t - Exp_t$$

$$GB_t = PB_t - r_{t-1}B_{t-1}$$

Then, the government budget constraint becomes:

$$B_t = B_{t-1} - GB_t$$

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Because of the presence of bonds, in order to avoid an explosive solution, similarly to Leeper (1991) and Anderson et al. (2013), a rule is introduced for lump-sum taxes:

\[
T_t = \eta T + (1 - \eta) \left[ \rho T_{t-1} + (1 - \rho)(GB_{t}^{\text{Target}} - GB_t) \right]
\]

Here, \( \eta \) shows the speed of adjustment; it is a 0-1 indicator; \( \eta = 1 \) if lump-sum taxes stay at the steady-state level and 0 otherwise. A similar rule is valid for all other fiscal instruments as well (value-added tax rate, personal income tax rate, employer and employee social security contributions and government consumption), and, all rules target total government balance \( GB_{t}^{\text{Target}} \).\(^{18}\)

The starting point of the pay-as-you-go plan is that the pension of each newly retired person depends on a pension-wage replacement rate, and on the individual’s previous wage stream:

\[
TR_{0,t}^{PG,YO} (i) = \nu_t IB_{b-1,t}^Y (i)
\]

\( TR_{t}^{PG,YO} (i) \) denotes pension received by individual \( i \) who just became retired in period \( t \).\(^{19} \) \( \nu_t \) is the gross pension-wage replacement rate (ratio of pension to gross wages), while \( IB_{b-1,t}^Y (i) \) is the individual’s previous years’ wage stream. To be more precise, it is a simple average of wages received in the previous \( Y \) years:

\[
IB_{b-1,t}^Y (i) = \frac{1}{Y} w_{t-1}^F L_{b-1,t-1}^F (i) + \frac{Y - 1}{Y} IB_{b-2,t-1}^Y (i)
\]

Then, total wage stream of all employees follows:

\[
IB_t^Y = \frac{1}{Y} w_{t-1}^F L_{t-1}^F + \frac{Y - 1}{Y} (1 - \omega_{t-2}) IB_{t-1}^Y
\]

Hence, the total pension of those old people who just retired is:

\[
TR_t^{PG,YO} = \nu_t \omega_{t-1} Y IB_{t-1}^Y
\]

Last, the total pension expenditure is the sum of the pension of the just-retired and of those who retired in any of the previous periods and are still alive:

\[
TR_t^{PG} = TR_t^{PG,YO} + (1 - \omega_{t-1}^O) TR_{t-1}^{PG}
\]

### 2.5.2 Fully funded pension system

The crucial difference between the two pension regimes is that in a fully funded regime \( TR_t^{PG} \) is no longer included in public expenditures:

\[
Exp_t = Gov_t + w_t^U U_t
\]

\(^{18}\)Calibration implies that \( GB^{\text{Target}} - GB = T \).

\(^{19}\)YO refers to just-retired, and stands for “young-old”.

Period-t social security contributions go directly to a pension fund, and future individual pension benefits are related to individual pension savings. The starting point is that individual \( i \) saves her social security contributions in a separate account:

\[
B_{b,t}^{Y,*}(i) = \Xi(\tau_{t}^{SSCW} + \tau_{t}^{SSCF})w_{t}^{F}L_{b,t}^{F}(i) + (1 + r_{t-1})B_{b-1,t-1}^{Y,*}(i)
\]

Here, \( B_{b,t}^{Y,*}(i) \) denotes individual \( i \)'s pension savings. As before, only employees and employers in the formal sector pay social security contributions; hence, informal income does not contribute to retirement savings.

Assuming that initial wealth is zero, total pension savings look like:

\[
B_{t}^{Y,*} = \Xi(\tau_{t}^{SSCW} + \tau_{t}^{SSCF})w_{t}^{F}L_{t}^{F} + (1 + r_{t-1})(1 - \omega_{t-1})B_{t-1}^{Y,*}
\]

When someone retires, an initial pension level is set by the government in that period based on her pension savings and the life expectancy. Later, the pension level will be adjusted by inflation only. It can be shown that \( TR_{0,t}^{FF,Y,O} \) which is a just-retired individual \( i \)'s pension in period \( t \) fulfills:

\[
(1 + r_{t-1})B_{b-1,t-1}^{Y,*}(i) = TR_{0,t}^{FF,Y,O}(i)\Omega_{t}^{O}
\]

Then, total pension expenditure of those who retired in period \( t \) \( (TR_{t}^{FF,Y,O}) \) and of all retired people \( (TR_{t}^{FF}) \), in the FF regime, are:

\[
(1 + r_{t-1})\omega_{t-1}B_{t-1}^{Y,*} = TR_{t}^{FF,Y,O}\Omega_{t}^{O}
\]

\[
TR_{t}^{FF} = TR_{t}^{FF,Y,O} + (1 - \omega_{t-1})TR_{t-1}^{FF}
\]

Last, as people get older, their pension savings shrink. So, the following holds:

\[
(1 + r_{t-1})\omega_{t-1}B_{t-1}^{Y,*} + (1 + r_{t-1})B_{t-1}^{O,*} = TR_{t}^{FF} + B_{t}^{O,*}
\]

where \( B_{t}^{O,*} \) is pension savings of previously retired people in a fully funded regime.

Finally, total pension savings are:

\[
B_{t}^{*} = B_{t}^{Y,*} + B_{t}^{O,*}
\]

### 2.6 Market clearing conditions

In equilibrium, all markets clear. First, the labor market clears.\(^{20}\)

The total number of workers in the formal and underground sector, and the number of unemployed people are:

\[^{20}\text{Also, the physical capital market clears: capital rented by firms equals capital produced by capital producers (this is why both are denoted by } K).\]
L_t^F = \sum_{b=0}^{\infty} L_{b,t}^F = \sum_{b=0}^{\infty} N_{b,t}^Y L_{b,t}^F(i)

L_t^I = \sum_{b=0}^{\infty} L_{b,t}^I = \sum_{b=0}^{\infty} N_{b,t}^Y L_{b,t}^I(i)

U_t = \sum_{b=0}^{\infty} U_{b,t} = \sum_{b=0}^{\infty} N_{b,t}^Y U_{b,t}(i)

Then, the total number of young people is equal to the sum of the number of people employed in either the formal or the informal sector and the number of unemployed people:

\[ U_t = N_t^Y - L_t^F - L_t^I \]

Total profits is the sum of sectoral profits:

\[ \text{Profit}_t = \text{profit}_t^F + \text{profit}_t^I \]

Then, there is an equilibrium in the assets market:

\[ B_t + Q_t^F K_t^F + Q_t^I K_t^I = B_t^Y + B_t^O + B_t^* \]

This means that the young and the retired save in risk-free bonds, and this aggregate savings is equal to the sum of government bonds and physical capital.

Regarding the goods market, in each sector, supply must equal the relevant demand and deadweight losses due to labor and product market frictions:

\[ Y_t^F = C_t^F + Inv_t + Gov_t + hc_t^F H_t^F + R \left( \frac{P_t^F}{P_{t-1}^F} \right) \]

\[ + Inv_t^F S \left( \frac{Inv_t^F}{Inv_{t-1}^F (1 + g_t)} \right) + Inv_t^I S \left( \frac{Inv_t^I}{Inv_{t-1}^I (1 + g_t)} \right) \]

\[ Y_t^I = C_t^I + hc_t^I H_t^I + R \left( \frac{P_t^I}{P_{t-1}^I} \right) \]

where

\[ C_t^F = C_t'^{O,F} + C_t'^{Y,F} \]

\[ C_t^I = C_t'^{O,I} + C_t'^{Y,I} \]

\[ Inv_t = Inv_t^F + Inv_t^I \]

Formal sector goods are either consumed by households or invested to produce physical capital. Also, the government purchases goods from formal firms. Hiring costs and price rigidity constitute deadweight losses, and there are also investment adjustment costs; these create a gap between production and demand. Concerning the informal side, the government does not buy informally produced goods and private investment only pertains to formal goods.
Finally, GDP is, as usual, the sum of household consumption, private investment and government consumption, where total consumption is the sum of formal and informal consumption:

\[ GDP_t = C_t + Inv_t + Gov_t \]
\[ C_t = C^F_t + p_t C^I_t \]

Also, production is the sum of formal and informal production:

\[ Y_t = Y^F_t + p_t Y^I_t \]

### 3 Impacts of aging

First and foremost, with a special focus on generational differences, we analyze how demographic changes affect the economy in our framework. We do that based on a parametrized version of the model. Parameters are shown in Tables 1 and 2; all of them are in the range of usual values of the literature. Because we parametrize the model (instead of calibrating or estimating it), no direct policy conclusions can be made concerning the magnitude of impulse responses. However, our aim is not to make such conclusions, but to describe the main channels of the model. All the simulations are made by Dynare 4.4.3 and the IRIS Toolbox\(^{21}\), and mean (permanent) deterministic simulations, moving from the original steady-state to a new long-run equilibrium.\(^{22}\)

We begin with the fact that if the mortality rate or the fertility rate decline, the relative size of the two generations changes. Specifically, there are relatively more old people and relatively fewer young people. This has labor market implications, both in the long- and in the short runs. To show this, we present impulse responses of a 5 pp permanent increase in the old-age dependency ratio over the next 30 years. In this section we consider the PAYG regime; in Section 4 we will address differences between the two regimes.

A decline in the fertility rate reduces long-run formal employment (Table 3). As the shrinking size of potential workers increases competition, formal wages go up. At the same time, informal employment goes up because some of the workers previously employed in the formal sector move to the shadow (as it provides higher wages than the value of unemployment benefits). Total employment drops, which is followed by a drop in output. Also, firms do not invest, so the capital stock shrinks. This is in line with a decline in savings which, in turn, is associated with an increase in consumption. Still, the young households’ consumption-to-income ratio goes down, because they finance the increase in public debt. The old’s consumption-to-income ratio increases because the higher formal wages also imply a rise in pensions, so the old consume more. Disinvestment is a gradual process; thus, in the short run, output goes up (as consumption increases) (Figures 1 and 2). Since the accumulation of public debt takes time, the young consumption-to-income ratio temporarily rises.

\(^{21}\)http://iristoolbox.codeplex.com

\(^{22}\)Our notes on parametrization and simulations also hold for the other sections of the paper.
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<tr>
<td>Government consumption expenditure to GDP ratio (%)</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>Unemployment benefit expenditure to GDP ratio (%)</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Gross pension-wage replacement rate (%)</td>
<td>$\nu$</td>
<td>85</td>
</tr>
<tr>
<td>Government debt as a share of GDP (%)</td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>Firing probability in the formal sector (%)</td>
<td>$p_{r_{EF}}$</td>
<td>10</td>
</tr>
<tr>
<td>Firing probability in the informal sector (%)</td>
<td>$p_{r_{Ei}}$</td>
<td>20</td>
</tr>
<tr>
<td>Ratio of hiring cost in wage in the formal sector</td>
<td></td>
<td>0.25</td>
</tr>
<tr>
<td>Ratio of hiring cost in wage in the informal sector</td>
<td></td>
<td>0.25</td>
</tr>
<tr>
<td>Bargaining power of workers in the formal sector</td>
<td>$\sigma_F$</td>
<td>0.75</td>
</tr>
<tr>
<td>Bargaining power of workers in the informal sector</td>
<td>$\sigma_I$</td>
<td>0.75</td>
</tr>
<tr>
<td>Elasticity of hiring cost wrt to hiring probability</td>
<td>$\alpha_{HC}$</td>
<td>0.5</td>
</tr>
<tr>
<td>Value added tax revenue in GDP (%)</td>
<td>$\tau_C$</td>
<td>7</td>
</tr>
<tr>
<td>Share of personal income tax revenue in GDP (%)</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>Share of employee SSC revenue in GDP (%)</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Share of employer SSC revenue in GDP (%)</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>Probability of becoming retired (%)</td>
<td>$\omega^Y$</td>
<td>1.5</td>
</tr>
<tr>
<td>Mortality rate of the retired (%)</td>
<td>$\omega^O$</td>
<td>5</td>
</tr>
<tr>
<td>Net fertility rate (%)</td>
<td>$n$</td>
<td>3</td>
</tr>
<tr>
<td>Technology growth rate (%)</td>
<td>$g_A$</td>
<td>1.5</td>
</tr>
<tr>
<td>Weight of informal goods in the utility function</td>
<td>$\chi$</td>
<td>0.75</td>
</tr>
<tr>
<td>Fraction of young households' profit and lump-sum taxes</td>
<td>$\xi_y$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Steady-state parameters
As regards the drop in the mortality rate, both formal and informal employment reduce. Also, as formal wages remain stable (thus keeping pensions stable) and workers can earn a higher wage in the formal than in the informal sector, some of them move out of the shadow. On the whole, again, total employment shrinks and so does output. However, contrary to the fertility decline, private consumption goes down because total (labor) income goes down. Both young and old people consume less, compared to their income, since they prepare themselves for a longer period of retirement. In the short run there is a rise in unemployment as more people start to enter the labor market and it takes time until they find jobs. Also, the young consumption-to-income ratio temporarily goes up, while that of the old immediately drops; the agents know that they will live longer, so the old start to prepare for the future.

Table 2: Dynamic parameters

<table>
<thead>
<tr>
<th>NAME</th>
<th>SIGN</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverse of intertemporal elasticity of substitution</td>
<td>γ</td>
<td>1</td>
</tr>
<tr>
<td>Adjustment cost of physical capital investment</td>
<td>φ₁</td>
<td>2.5</td>
</tr>
<tr>
<td>Autoregressive parameter of no-pension policies</td>
<td>θ₂</td>
<td>0.52</td>
</tr>
<tr>
<td>Autoregressive parameter of lump-sum tax policy</td>
<td>θ₃</td>
<td>0.9</td>
</tr>
<tr>
<td>Autoregressive parameter of pension policies</td>
<td>θ₄</td>
<td>0.99</td>
</tr>
<tr>
<td>Autoregressive parameter of pension regime switch</td>
<td>θ₅</td>
<td>0.95</td>
</tr>
<tr>
<td>Rotemberg price adjustment cost</td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>Price indexation</td>
<td>γₚ</td>
<td>0.75</td>
</tr>
<tr>
<td>Interest rate smoothing</td>
<td>ρᵢ</td>
<td>0.5</td>
</tr>
<tr>
<td>Reaction to inflation in Taylor rule</td>
<td>ρₓ</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3: Long-run (100-year) impacts of aging

<table>
<thead>
<tr>
<th>Dependency ratio increase with</th>
<th>GDP per capita (%)</th>
<th>Total household consumption per capita (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PAYG</td>
<td>FF</td>
</tr>
<tr>
<td>Lower mortality rate</td>
<td>-2.8</td>
<td>-2.6</td>
</tr>
<tr>
<td>Lower fertility rate</td>
<td>-2.6</td>
<td>-2.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dependency ratio increase with</th>
<th>Young consumption-income ratio (pp)</th>
<th>Old consumption-income ratio (pp)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PAYG</td>
<td>FF</td>
</tr>
<tr>
<td>Lower mortality rate</td>
<td>-3.4</td>
<td>-1.5</td>
</tr>
<tr>
<td>Lower fertility rate</td>
<td>-5.1</td>
<td>10.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dependency ratio increase with</th>
<th>Formal employment (%)</th>
<th>Informal employment (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PAYG</td>
<td>FF</td>
</tr>
<tr>
<td>Lower mortality rate</td>
<td>-2.5</td>
<td>-2.4</td>
</tr>
<tr>
<td>Lower fertility rate</td>
<td>-4.1</td>
<td>-4.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dependency ratio increase with</th>
<th>Unemployment rate (pp)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PAYG</td>
</tr>
<tr>
<td>Lower mortality rate</td>
<td>0.0</td>
</tr>
<tr>
<td>Lower fertility rate</td>
<td>-1.4</td>
</tr>
</tbody>
</table>
Figure 1: Short-run (10-year) impacts of a drop in mortality rate
Figure 2: Short-run (10-year) impacts of a drop in fertility rate
4 Impacts of public policies

We now turn our attention towards fiscal multipliers. Aging decreases output and increases public debt, which prompts some response from the government. Table 4 shows the long run, while Figures 3 and 4 present the short-run reactions. The multipliers of no-pension policies are directly comparable as they are all 1 per cent of steady-state GDP size. Specifically, we investigate (abbreviations on charts are in brackets): i) a reduction in personal income tax revenue (T), ii) a drop in employees’ social security contributions (SSCE), iii) a drop in employers’ social security contributions (SSCR), iv) a value-added tax revenue cut (VAT) and v) a cut in government consumption expenditure (G). Additionally, we examine a change in the retirement probability (Ret. age) (which is a proxy for modifying the retirement age) and the pension-wage replacement rate (Rep. rate). The probability of retirement is increased by 0.04 pp, while the replacement rate is raised by 2.5 pp; these shocks are calibrated such that they equal 1 per cent of medium-term GDP size. It is important to note that the budgetary impacts of these two latter policies differ in size, so they are not directly comparable as regards the magnitudes of impulse responses. Last, all policies are permanent.

First, cutting labor income tax revenues implies a slight decline in long-run unemployment. This is true for both employee and employer taxes. The underlying reason for this is that the relative price of labor and capital changes, that is, labor becomes less expensive, changing firms’ incentives. As a result, employment goes up and so does demand, namely, both household consumption and GDP.

Other policies, however, do not affect long-term unemployment as they do not directly influence the production side. Also, in contrast to labor income tax cuts, a lower retirement age results in lower consumption and GDP. The reason for this is that more people retire, which results in less income for them (because pensions are usually lower than wages), so consumption and demand in general shrink. Furthermore, even though fiscal expansion via VAT is almost output-neutral, because the relative price of consumption goes down, private consumption increases. Additionally, the larger the public consumption, the lower the private investment, which is the usual crowding-out impact. At the same time, a cut in VAT implies a rise in household savings and so investment to finance the rise in public debt.

Fiscal expansion always implies a drop in the share of young agents’ consumption in total consumption, especially policies those which mainly affect the demand side of the economy. The reason for this is that the young typically finance the increase in public debt caused by fiscal expansion.

With the parameter values we picked, the size of the shadow economy does not significantly change in the long run. Yet, this finding might be country-specific. Indeed, Baksa and Munkacsi (2016) find, based on Portuguese and Spanish data, that value-added tax, government consumption and sometimes even labor income tax policies have an impact on long-run informality.
Table 4: Long-run (100-year) effects of fiscal expansion in PAYG and fully funded regimes

<table>
<thead>
<tr>
<th>Fiscal expansion with</th>
<th>GDP per capita (%)</th>
<th>Unemployment rate (% point)</th>
<th>Total household consumption per capita (%)</th>
<th>Share of informal employment in total employment (% point)</th>
<th>Share of formal GDP in total GDP (% point)</th>
<th>Government consumption exp./GDP</th>
<th>Fiscal expansion with</th>
<th>GDP per capita (%)</th>
<th>Unemployment rate (% point)</th>
<th>Total household consumption per capita (%)</th>
<th>Share of informal employment in total employment (% point)</th>
<th>Share of formal GDP in total GDP (% point)</th>
<th>Government consumption exp./GDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>PayG</td>
<td>1.2</td>
<td>-0.4</td>
<td>1.7</td>
<td>-1.7</td>
<td>0.1</td>
<td>0.0</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Fully funded</td>
<td>1.2</td>
<td>-0.4</td>
<td>1.7</td>
<td>-1.7</td>
<td>0.1</td>
<td>0.0</td>
<td>0.1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Figure 3: Short-run (10-year) multipliers of no-pension policies in a PAYG regime
Figure 4: Short-run (10-year) multipliers of pension policies in a PAYG regime
The main difference between the short- and long-run reactions is that in the short run, cutting VAT revenues or raising government consumption increases formal GDP and decreases informal employment; the higher demand requires more workers, who arrive from the informal sector. Nevertheless, without a pension regime switch, the workers’ incentives remain the same; hence, the impact is only temporary. Baksa and Munkacsi (2016) show that a pension regime switch causes not only a short-, but also a long-term reduction in the size of the underground sector. In the short-run, the main driving force is the presence of rigidities, which induces the labor markets to accommodate when relative prices change. Furthermore, there is also a short-term shift in consumption towards the young generations if labor income taxes are cut. This is a direct consequence of the fact that lower labor income taxes increase the disposable income of the young. Still, because the young finance the rising public debt by saving more, in the long run this temporary effect fades away.

5 Comparing PAYG and fully funded regimes

Pensions are calculated and financed differently in PAYG plans, on the one hand, and fully funded plans, on the other. In the former, pension benefits depend on the previous wage stream and a pension-wage replacement rate. In the latter, by contrast, an individual worker’s contributions are collected in a separate account and are used to finance pension benefits of the account owner. Hence, workers have different incentives in the two regimes, and the path of public debt also differs between them. In the next two subsections we compare the macroeconomic impacts of aging and public policies within the two pension regimes.

5.1 Aging

Table 3 and Figures 1 and 2 do not only show the macroeconomic reactions in a PAYG plan, but also those in a fully funded regime.

Regarding the fertility reduction, the main difference between the two regimes is that in the long term public debt increases less in a fully funded plan. This is so because interest payments on the debt service are lower (because the interest rate itself is lower as the regime is more sustainable, i.e. there is pension wealth). Hence, young households need to finance less public debt, which implies that they save less and consume more; their consumption-to-income ratio increases. At the same time, the old consumption-to-income ratio goes down, in contrast to the PAYG plan, because consumption goes up, but pensions go up even more.

When the mortality rate declines, again, because in a fully funded regime public debt goes up less in the long-run, the young save less and consume more, so the young consumption-to-income ratio declines less. The old, however, consume more relative to their income as their income significantly increases (their pension goes down because they live longer, but they save more as they know that they will live longer). In the short run, again because the agents know that they will live longer, and in contrast to the pay-as-you-go regime, already the young start to save. Hence, their consumption goes down relative to their income.
5.2 Public policies

In the long run there are only minor differences between the macroeconomic reactions within the two pension regimes (Table 4). One obvious exception is that in a fully funded regime lowering employee or employer social security contribution rates does not affect the public debt level.

As regards the short term, however, there is a crucial distinction between the two plans. Specifically, the composition of private consumption changes in different ways (Figure 5). In a fully funded regime, the young temporarily consume relatively more than the old. If taxes are reduced or the government spends more, in both regimes, the young’s disposable income goes up. However, liquidity is larger in the FF regime (due to the presence of pension wealth), so the interest rate is lower. The lower interest rate implies that public debt increases less because the interest payments are lower. Hence, workers can consume more as they need to finance a lower debt service.

6 The role of informality and unemployment

The inclusion of a shadow economy into a Gertler-type overlapping generations model constitutes a vital novelty. On the one hand, informality means that taxes are avoided in the shadow; on the other hand, the level of regulation is lower there (Table 5). As noted in the introduction, we believe that the shadow economy can reshape macroeconomic responses to fiscal policies. The same is true for unemployment (at least in the short run, because of the presence of rigidities). Additionally, there is an obvious advantage of taking them into account, namely, that we can study how they respond to various policies.
Thus, we take two alternative scenarios with the baseline scenario presented so far, namely, a model without the shadow economy but with unemployment and a model without the shadow economy and without unemployment. Here, we present our findings with a PAYG plan.

Regarding the long run, both the presence of informality and unemployment are crucial for private consumption and GDP; without them the multipliers are much lower, but only for labor income tax policies (Table 6). This is so because if the shadow economy is neglected, the tax base is larger (some part of informality is treated as non-shadow). Hence, for example, a lower personal income tax reduction is enough to induce the same size shock.

The short term presents a different view. Here, unemployment is also affected and so is the share of young consumption in total consumption (Figures 6-8). Particularly, if labor income taxes are reduced, unemployment increases less on impact if the shadow economy is neglected. Also, the share of young consumption increases less. The intuition is that without a shadow economy those who leave formal employment are considered to be unemployed rather than moving into the shadow.

Additionally, without an underground sector, a value-added tax cut or expansion of government consumption does not significantly modify the unemployment rate. Furthermore, the reactions of GDP and private consumption become less pronounced without a shadow sector or unemployment, but the signs also differ: the negative signs might become positive. This can be traced back to the unemployment response described above.

---

Table 5: Rigidities in the formal and informal sectors of the model

<table>
<thead>
<tr>
<th>Hiring costs</th>
<th>Formal sector</th>
<th>Informal sector</th>
</tr>
</thead>
<tbody>
<tr>
<td>Workers' bargaining power over wages</td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>Firing probability</td>
<td>Low</td>
<td>High</td>
</tr>
<tr>
<td>Taxation</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Government consumption</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

---

23 Some model parameters are modified in order to disregard informality. These are the weight on informal household consumption in the utility function, the firing probability, the hiring cost parameter and the share of capital income in the production function (all in the informal sector); the first of these is reduced, while all the others are increased. Then, in the model without unemployment, the formal hiring cost parameter is lowered.

24 Very similar conclusions can be made for the fully funded regime; results are available from the authors upon request.
Table 6: Long-run (100-year) effects of fiscal expansion in a PAYG regime without shadow economy and unemployment

<table>
<thead>
<tr>
<th>Fiscal expansion with</th>
<th>GDP per capita (%)</th>
<th>Unemployment rate (%point)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Baseline</td>
<td>No shadow economy</td>
</tr>
<tr>
<td>Personal income tax</td>
<td>1.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Employee SSC</td>
<td>1.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Employer SSC</td>
<td>0.9</td>
<td>0.1</td>
</tr>
<tr>
<td>Pension-wage replacement rate</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Retirement probability</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
<tr>
<td>Value added tax</td>
<td>0.0</td>
<td>-0.1</td>
</tr>
<tr>
<td>Government consumption exp./GDP</td>
<td>0.1</td>
<td>-0.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Fiscal expansion with</th>
<th>Total household consumption per capita (%)</th>
<th>Share of young household consumption in total consumption (%point)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Baseline</td>
<td>No shadow economy</td>
</tr>
<tr>
<td>Personal income tax</td>
<td>1.7</td>
<td>0.3</td>
</tr>
<tr>
<td>Employee SSC</td>
<td>1.7</td>
<td>0.3</td>
</tr>
<tr>
<td>Employer SSC</td>
<td>1.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Pension-wage replacement rate</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Retirement probability</td>
<td>-0.7</td>
<td>-0.7</td>
</tr>
<tr>
<td>Value added tax</td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>Government consumption exp./GDP</td>
<td>-1.7</td>
<td>-1.9</td>
</tr>
</tbody>
</table>
Figure 6: Short-run (10-year) multipliers of labor policies without shadow economy and unemployment in a PAYG regime
Figure 7: Short-run (10-year) multipliers of demand policies without shadow economy and unemployment in a PAYG regime
Figure 8: Short-run (10-year) multipliers of pension policies without shadow economy and unemployment in a PAYG regime
7 Summary

In this paper we presented OGRE, a dynamic general equilibrium model with overlapping generations and demography, unemployment and wage bargaining, and an underground sector. OGRE was developed to investigate the macroeconomic effects of a wide range of public policies in what might be called the “Age of Aging.”

The main novelty was the inclusion of an underground sector. To the best of our knowledge, our model is the first Gertler-type OLG model with informality. Additionally, we believe that, besides Berger et al. (2009), OGRE is the only Blanchard-Yaari-type OLG model that takes into account unemployment. Moreover, OGRE, is suitable for investigating the impacts of retirement age changes, and those of both reforming pension regimes and pension regime switches, thus contributing even further to the literature.

The main channel of aging in the model is that the economy’s size shrinks, implying that employment and thus demand are also negatively affected. Short- and long-run implications of different demographic shocks differ, as do those of the public policies. Here, a central point is that labor income tax cuts imply a rise in the unemployment rate while other policies do not distort the labor market in the long run. Short-run costs exist, including, for example, a temporary rise in unemployment after a cut in VAT or an increase in government consumption.

Workers’ incentives are different in the PAYG plan, on the one hand, and the FF pension plan, on the other. Hence, the paths of public debt are also different in the two regimes. We find that the long-run macroeconomic responses of fiscal policies do not significantly differ across the two regimes. In the short run, however, the composition of private consumption reacts differently, namely, there is often a shift towards young generations in a fully funded pension regime. As regards aging, the composition of consumption reacts differently in different pension regimes in the long run as well.

As regards the role of unemployment and informality, for labor income tax policies their presence was crucial concerning household consumption or GDP; without them the multipliers were much lower. Without a shadow economy, the tax base is larger; thus, a lower tax cut induces the same size shock. In the short term, unemployment responses also changed and so did the composition of consumption. The intuition is that without informality those who leave formal employment are considered to be unemployed instead of moving into informality. Even the signs of some multipliers reversed, not only the magnitudes were influenced. This means that without these modeling features the short-term reactions might be biased, leading to incorrect policy conclusions.

We are convinced that the inclusion of a shadow economy in an OLG framework was a vital step in studying the fiscal consequences of aging. Nonetheless, our framework lacks some important features. First, OGRE is a closed economy, but there are international spillovers. Second, as inside the cohorts there was no heterogeneity, inequality could not be investigated. The elements of an open economy and inequality would constitute excellent lines of future research.
Appendix

A Overlapping generations’ optimization

First, we focus on solving the optimizing problems of the young and old generations. Then, we describe in detail pay-as-you-go and fully funded pension systems. At the end, we list all the normalized equations, i.e. equations detrended by technology and population growth, provide the steady state calculation of the model and show evidence of the model’s robustness.25

A.1 Demography

Total population \((N_t)\) is equal to the sum of the number of old (retired) \((N^O_t)\) and young (worker) people \((N^Y_t)\):

\[
N_t = N^O_t + N^Y_t
\]

\[
N^Y_t = (1 - \omega^Y_{t-1})N^Y_{t-1} + n_tN^Y_{t-1}
\]

\[
N^O_t = (1 - \omega^O_{t-1})N^O_{t-1} + \omega^Y_{t-1}N^Y_{t-1}
\]

Like most of the general equilibrium models, we focus on the relative shares and not on the levels. \(s_t\) denotes the ratio of the number of old and young people, while \(s^Y_t\) denotes the share of young people in the whole population:

\[
s_t = \frac{N^O_t}{N^Y_t} = \frac{(1 - \omega^O_{t-1})N^O_{t-1} + \omega^Y_{t-1}N^Y_{t-1}}{N^Y_{t-1}} = (1 - \omega^O_{t-1}) \frac{N^O_{t-1}N^Y_{t-1}}{N^Y_{t-1}} + \\
\hspace{1cm} + \omega^Y_{t-1} \frac{N^Y_{t-1}}{N^Y_t} = \frac{(1 - \omega^O_{t-1})}{(1 - \omega^Y_{t-1} + n_t)}s_{t-1} + \frac{\omega^Y_{t-1}}{(1 - \omega^Y_{t-1} + n_t)}
\]

\[
s^Y_t = \frac{N^Y_t}{N_t} = \frac{N^Y_t}{N^Y_t + N^O_t} = \frac{1}{1 + \frac{N^O_t}{N^Y_t}} = \frac{1}{1 + s_t}
\]

Then, we can express the growth rate of each cohort:

\[
1 + g_t^{N,Y} = \frac{N^Y_t}{N^Y_{t-1}} = \frac{(1 - \omega^Y_{t-1})N^Y_{t-1} + n_tN^Y_{t-1}}{N^Y_{t-1}} = 1 - \omega^Y_{t-1} + n_t
\]

\[
1 + g_t^{N,O} = \frac{N^O_t}{N^O_{t-1}} = \frac{(1 - \omega^O_{t-1})N^O_{t-1} + \omega^Y_{t-1}N^Y_{t-1}}{N^O_{t-1}} = (1 - \omega^O_{t-1}) + \frac{\omega^Y_{t-1}}{s_{t-1}}
\]

Finally, population growth follows as:

\[
1 + g_t^{N} = \frac{N^Y_t}{N^Y_{t-1}} + \frac{N^O_t}{N^O_{t-1}} = \frac{N^Y_{t-1} + N^O_{t-1}}{N^Y_{t-1} + N^O_{t-1}} = \frac{1 + g_t^{N,Y} + g_t^{N,O}}{1 + s_{t-1}} = \\
\hspace{1cm} \frac{1 + g_t^{N,Y} + \frac{N^O_{t-1}N^Y_t}{N^Y_{t-1}}}{1 + s_{t-1}} = \frac{1 + g_t^{N,Y} + s_t(1 + g_t^{N,Y})}{1 + s_{t-1}} = (1 + g_t^{N,Y}) \frac{1 + s_t}{1 + s_{t-1}}
\]

25Regarding any other details, further information is available from the authors upon request.
A.2 Retired generation

First-order conditions of a retired agent

Retired agent $i$ of retired cohort $a$ is one individual who retired $a$ years ago. Each agent maximises the following Bellman equation:

$$V^O(B^O_a, t(i)) = \max \left\{ (1 + \epsilon^C_t) \left[ \frac{1}{1 - \gamma} \{ C_{a,t}^{O,F}(i) \}^{-\gamma} + \frac{\chi}{1 - \gamma} \{ C_{a,t}^{O,I}(i) \}^{-\gamma} \right] + \beta E_t(1 - \omega^O_t)V^O(B^O_a, t(i)) \right\}$$

subject to this budget constraint:

$$(1 + \tau^C_t)C_{a,t}^{O,F}(i) + p^I_t C_{a,t}^{O,I}(i) + (1 - \omega^O_t)B^O_{a,t}(i) = (1 + r_{t-1})B^O_{a-1,t-1}(i) + TR^{PG,YO}_{a,t}(i) + TR^{FF,YO}_{a,t}(i) + Profit^O_{a,t}(i) - T^O_{a,t}(i)$$

First-order conditions:

$$C_{a,t}^{O,F}(i) : (1 + \epsilon^C_t) \left[ C_{a,t}^{O,F}(i) \right]^{-\gamma} + \lambda^O_{a,t} (1 + \tau^C_t) = 0$$

$$C_{a,t}^{O,I}(i) : (1 + \epsilon^C_t) \chi \left[ C_{a,t}^{O,I}(i) \right]^{-\gamma} + \lambda^O_{a,t} p^I_t = 0$$

$$B^O_{a,t}(i) : \beta E_t(1 - \omega^O_t)V^O_{B^O_a, t(i)} + E_t(1 - \omega^O_t)\lambda^O_{a,t} = 0$$

One-period-ahead Envelope theorem:

$$E_t V^O_{B^O_a, t(i)} = -E_t \lambda^O_{a+1,t+1}(1 + r_t)$$

The first-order conditions imply the Euler-equation:

$$\beta E_t \left[ (1 + \epsilon^C_{t+1}) \left( C_{a,t}^{O,F}(i) \right)^{-\gamma} (1 + r_t) \right] \left[ (1 + \tau^C_t) \right] \frac{1 + \tau^C_{t+1}}{1 + \tau^C_{t+1}} = 1$$

which can be rearranged:

$$C_{a,t}^{O,F}(i) E_t \left\{ \beta \frac{1 + \epsilon^C_{t+1}}{1 + \epsilon^C_{t+1}} (1 + r_t) \frac{1 + \tau^C_{t+1}}{1 + \tau^C_{t+1}} \right\} \frac{1}{1 + \tau^C_{t+1}} = E_t C_{a+1,t+1}^{O,F}$$

$$E_t C_{a+1,t+1}(i) = E_t C_{a,t}^{O,F}(i) (1 + r_t) \frac{1}{1 + \tau^C_{t+1}} = \Lambda_{t+1}$$

where $E_t \Lambda_{t+1} = E_t \left\{ \frac{1 + \epsilon^C_{t+1}}{1 + \epsilon^C_{t+1}} \frac{1 + \tau^C_{t+1}}{1 + \tau^C_{t+1}} \right\} \frac{1}{1 + \tau^C_{t+1}}$.

Based on the Euler-equation all future retired consumptions follow:

$$E_t C_{a+n,t+n}^{O,F}(i) = C_{a,t}^{O,F}(i) E_t \prod_{k=1}^{n} (1 + r_{t+k-1}) \frac{1}{1 + \tau^C_{t+k}} \Lambda_{t+k}$$
Then, the substitution between formal and informal goods also follows from the first-order conditions:

\[
(1 + \varepsilon^C) \{ C_{a,t}^{O,F} (i) \}^{-\gamma} = (1 + \varepsilon^C) \chi \{ C_{a,t}^{O,I} (i) \}^{-\gamma} \frac{1 + \tau^C}{p_t}
\]

which can be rewritten as:

\[
C_{a,t}^{O,I} (i) = \Upsilon_t C_{a,t}^{O,F} (i)
\]

where \( \Upsilon_t = \left\{ \frac{\chi^{1+\tau^C}}{p_t^\gamma} \right\} \).

**Individual consumption of a retired agent**

First, we derive the intertemporal budget constraint from the one-period budget constraint:

\[
E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^G_{t+k-1}) \left( (1 + \tau^C_{t+n}) C_{a+n,t+n}^{O,F} + p^I_{t+n} C_{a+n,t+n}^{O,I} \right) =
\]

\[
E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^G_{t+k-1}) \left( TR_{a+n,t+n}^{PG,YO} + TR_{a+n,t+n}^{FF,YO} + \text{Profit}_{a+n,t+n}^O - T_{a+n,t+n}^{O} \right) + (1 + r_{t-1}) B_{a-1,t-1}^{O}(i)
\]

if \( k > n \) and \( r_{t+k} = 0 \).

Then, we plug in the formal-informal substitution equation:

\[
E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^G_{t+k-1}) \left( (1 + \tau^C_{t+n}) C_{a+n,t+n}^{O,F} + p^I_{t+n} C_{a+n,t+n}^{O,F} \Upsilon_{t+n} \right) =
\]

\[
E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^G_{t+k-1}) \left( TR_{a+n,t+n}^{PG,YO} + TR_{a+n,t+n}^{FF,YO} + \text{Profit}_{a+n,t+n}^O - T_{a+n,t+n}^{O} \right) + (1 + r_{t-1}) B_{a-1,t-1}^{O}(i)
\]

After some rearranging:

\[
E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^G_{t+k-1}) C_{a+n,t+n}^{O,F} \left( (1 + \tau^C_{t+n}) + p^I_{t+n} \Upsilon_{t+n} \right) =
\]

\[
E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^G_{t+k-1}) \left( TR_{a+n,t+n}^{PG,YO} + TR_{a+n,t+n}^{FF,YO} + \text{Profit}_{a+n,t+n}^O - T_{a+n,t+n}^{O} \right) + (1 + r_{t-1}) B_{a-1,t-1}^{O}(i)
\]
Now, we can use the Euler equation for future consumptions:

\[
E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega_{t+k-1}^{O}) C_{a,t}^{O,F}(i) \prod_{k=1}^{n} (1 + r_{t+k-1})^{\frac{1}{t}} \Lambda_{t+k} ((1 + \tau_{t+k}^{C}) + p_{t+n} Y_{t+n}) =
\]

\[
= E_t \sum_{n=0}^{\infty} (1 - \omega_{t+k-1}^{O}) \left( TR_{a+n,t+n}^{PG,YO}(i) + TR_{a+n,t+n}^{YY,F,FO}(i) + Profit_{a+n,t+n}(i) - T_{a+n,t+n}(i) \right) +
\]

\[+(1 + r_{t-1}) B_{a-1,t-1}^{O}(i) \]

Finally, if we rearrange we get consumption of agent \( i \) of cohort \( a \) at time \( t \) as a function of present value of pension and other income and initial wealth:

\[
\bar{C}_{a,t}^{O,F}(i) = \frac{E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega_{t+k-1}^{O}) \left( TR_{a+n,t+n}^{PG,YO}(i) + TR_{a+n,t+n}^{YY,F,FO}(i) + Profit_{a+n,t+n}(i) - T_{a+n,t+n}(i) \right) +
\]

\[+(1 + r_{t-1}) B_{a-1,t-1}^{O}(i) \]

After some simplification and by introducing some additional variables, the final version of consumption of agent \( i \) of cohort \( a \) at time \( t \) is:

\[
\bar{H}_{t}^{O} \bar{C}_{a,t}^{O,F}(i) = ( TR_{a,t}^{PG,YO}(i) + TR_{a,t}^{YY,F,FO}(i) ) \bar{\Omega}_{t}^{O} + \bar{T}_{a,t}(i) + (1 + r_{t-1}) B_{a-1,t-1}^{O}(i)
\]

\[
\bar{\Omega}_{t}^{O} = 1 + E_t \frac{1 - \omega_{t}^{O}}{1 + r_{t}} \bar{\Omega}_{t+1}^{O}
\]

\[
\bar{T}_{a,t}(i) = Profit_{a,t}(i) - T_{a,t}(i) + E_t \frac{1 - \omega_{t}^{O}}{1 + r_{t}} \bar{T}_{a+1,t+1}(i)
\]

\[
\bar{H}_{t}^{O} = (1 + \tau_{t}^{C}) + p_{t} Y_{t} + E_t (1 - \omega_{t}^{O}) (1 + r_{t})^{\frac{1}{t}} \Lambda_{t+1} \bar{H}_{t+1}^{O}
\]

Here, we would like to note that \( TR_{a,t}^{PG,YO}(i) = TR_{0,t}^{PG,YO}(i) \forall n > 0 \), and the same is true for fully funded pensions.

**Aggregate consumption of the retired cohort**

Aggregate consumption is equal to the sum of total pension and other income and initial wealth:

\[
\sum_{a=0}^{\infty} N_{a,t}^{O} C_{a,t}^{O,F}(i) \bar{H}_{t}^{O} = \sum_{a=0}^{\infty} N_{a,t}^{O} \left( TR_{a,t}^{PG,YO}(i) + TR_{a,t}^{YY,F,FO}(i) \right) \bar{\Omega}_{t}^{O} +
\]

\[+ \sum_{a=0}^{\infty} N_{a,t}^{O} T_{a,t}(i) + (1 + r_{t-1}) \sum_{a=0}^{\infty} N_{a,t}^{O} B_{a-1,t-1}^{O}(i) \]

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First, the number of old people declines over time:

\[ N_{a+1,t}^O = (1 - \omega^O_{t-1})N_{a,t-1}^O \]
\[ N_{a+2,t}^O = (1 - \omega^O_{t-1})(1 - \omega^O_{t-2})N_{a,t-2}^O \]

and

\[ N_t^O = \sum_{a=0}^{\infty} N_{a,t}^O \]

Now, we can express aggregate pension income in period \( t \) of those who retired today, one period before, etc.:

\[ TR_{t}^{PG,YO} + TR_{t}^{FF,YO} = N_{0,t}(TR_{0,t}^{PG,YO}(i) + TR_{0,t}^{FF,YO}(i)) \]
\[ (1 - \omega^O_{t-1})(TR_{t-1}^{PG,YO} + TR_{t-1}^{FF,YO}) = (1 - \omega^O_{t-1})N_{0,t-1}(TR_{0,t-1}^{PG,YO}(i) + TR_{0,t-1}^{FF,YO}(i)) = \]
\[ = N_{1,t}(TR_{1,t}^{PG,YO}(i) + TR_{1,t}^{FF,YO}(i)) \]

using \( TR_{n,t+n}^{PG,YO}(i) = TR_{0,t}^{PG,YO}(i) \) \( \forall n > 0 \) again.

Then, adding up all pensions implies:

\[ TR_{t}^{PG} + TR_{t}^{FF} = TR_{t}^{PG,YO} + TR_{t}^{FF,YO} + (1 - \omega^O_{t-1})(TR_{t-1}^{PG,YO} + TR_{t-1}^{FF,YO}) + \]
\[ +(1 - \omega^O_{t-1})(1 - \omega^O_{t-2})(TR_{t-2}^{PG,YO} + TR_{t-2}^{FF,YO}) + ... \]

Similarly, we can express the aggregate value of all other incomes:

\[ I_t^O = \sum_{a=0}^{\infty} N_{a,t}^O I_{a,t}(i) = \sum_{a=0}^{\infty} N_{a,t}^O (Profit_{a,t}^O(i) - T_{a,t}^O(i)) + \]
\[ +E_t \frac{1 - \omega^O_t}{1 + r_t} \sum_{a=0}^{\infty} N_{a,t}^O I_{a+1,t+1}(i) = \]
\[ = Profit_t^O - T_t^O + E_t \frac{1}{1 + r_t} \sum_{a=0}^{\infty} N_{a+1,t+1}^O I_{a+1,t+1}(i) \]

Where we note, because \( I_t^O \) refers to those who are already retired in period \( t \), that

\[ E_t \sum_{a=0}^{\infty} N_{a+1,t+1}^O I_{a+1,t+1}(i) = E_t I_{t+1}^O - E_t N_{a,t+1}^O I_{a,t+1}^O(i) \]

Plugging this back to the recursive formula results in:

\[ I_t^O = Profit_t^O - T_t^O + E_t \frac{1}{1 + r_t} (I_{t+1}^O - N_{a,t+1}^O I_{a,t+1}(i)) \]
Then, the law of large numbers implies:

\[ T_t^O = Profit_t^O - T_t^O + E_t \frac{1}{1 + r_t} T_{t+1}^O \left( 1 - \frac{N_{a,t+1}^{O}}{N_{t+1}^{O}} \right) \]

We rearrange the last term:

\[
1 - E_t \frac{N_{a,t+1}^{O}}{N_{t+1}^{O}} = 1 - E_t \frac{\omega_t Y N_Y}{N_{t+1}^{O}} = 1 - E_t \frac{\omega_t Y N_{t}^{O}}{N_{t}^{O}} =
\]

\[
= 1 - E_t \frac{\omega_t Y}{s_t} \frac{1}{1 + g_{t+1}^{N,O}} = 1 - E_t \frac{\omega_t Y}{s_t} \frac{1}{1 - \omega_t^O + \frac{\omega_Y}{s_t}} = 1 - E_t \frac{\omega_t Y}{1 - \omega_t^O + \frac{\omega_Y}{s_t}} =
\]

\[
E_t \frac{1 - \omega_t^O + \frac{\omega_Y}{s_t} - \omega_t^O}{1 - \omega_t^O + \frac{\omega_Y}{s_t}} = E_t \frac{1}{1 - \omega_t^O + \frac{\omega_Y}{s_t}} = E_t \frac{1 - \omega_t^O}{1 + g_{t+1}^{N,O}}
\]

Finally, total other income is:

\[ T_t^O = Profit_t^O - T_t^O + E_t \frac{1 - \omega_t^O}{(1 + r_t)(1 + g_{t+1}^{N,O})} T_{t+1}^O \]

where we can define a rule how to divide the aggregate profits and lump-sum taxes among the retired and young cohorts:

\[ Profit_t^O - T_t^O = (1 - \xi)(Profit_t - T_t) \]

and \( \xi \) is a parameter between zero and one, that shows the fraction of profit minus lump-sum taxes that goes to young cohort.

Now, aggregate consumption of the retired cohort is defined as:

\[ C_{t}^{O,F} = \sum_{a=0}^{\infty} N_{a,t}^{O} C_{a,t}^{O,F}(i) \]

while total savings of the retired is:

\[ \sum_{a=0}^{\infty} N_{a,t}^{O} B_{a,t-1}(i) = N_{0,t}^{O} B_{0,t-1}(i) + \sum_{a=1}^{\infty} N_{a,t}^{O} B_{a,t-1}(i) \]

Here, we need to be careful with the just-retired agents, who were young one period before and had different savings then. We can use the law of large numbers, and the following expression: \( N_{0,t}^{O} = \omega_{t-1}^Y N_{t-1}^Y \):

\[ N_{0,t}^{O} B_{0,t-1}(i) = N_{0,t}^{O} \sum_{b=1}^{\infty} B_{b,t-1}(i) \simeq \omega_{t-1}^Y N_{t-1}^Y \frac{B_{t-1}^{Y,\text{last}}}{N_{t-1}^{Y}} \]

where the \textit{last} refers to the fact those who retired today spent their last year in the young cohort during the previous year.
Then, from \( t - 1 \) to \( t \) it is easy to see that:

\[
\sum_{a=1}^{\infty} N_{a,t}^O = \sum_{a=1}^{\infty} (1 - \omega_{t-1}^O) N_{a-1,t-1}^O
\]

which implies that

\[
\sum_{a=0}^{\infty} N_{a,t}^O B_{a,t-1}^O(i) = \omega_{t-1}^Y B_{t-1}^Y + \sum_{a=1}^{\infty} (1 - \omega_{t-1}^O) N_{a-1,t-1}^O B_{a,t-1}^O(i)
\]

Here, the second term means that only those retired agents accumulate savings who expect to survive the next period. Hence, the amount of aggregate old-age savings from the previous period is \( B_{t-1}^O = \sum_{a=0}^{\infty} (1 - \omega_{t-1}^O) N_{a-1,t-1}^O B_{a,t-1}^O(i) \). Then, overall savings of the retired cohort in period \( t \) can be expressed easily by adding just-retired savings from the previous period’s young cohorts:

\[
\sum_{a=0}^{\infty} N_{a,t}^O B_{a,t-1}^O(i) = \omega_{t-1}^Y B_{t-1}^Y + B_{t-1}^O
\]

As a last step, we combine all parts of the equation, so, the aggregate consumption of formal goods of the retired cohort is:

\[
\mathcal{H}_t^O C_t^{O,F} = (TR_t^{PG} + TR_t^{FF}) \Omega_t^O + T_t^O + (1 + r_{t-1})\left(\omega_{t-1}^Y B_{t-1}^Y + B_{t-1}^O\right)
\]

Also, informal consumption is:

\[
C_t^{O,I} = \Upsilon_t C_t^{O,F}
\]

### A.3 Young generation

**First-order conditions of a young agent**

Young agent \( i \) of young cohort \( b \) is one individual of its cohort who started to work (i.e., was born) \( b \) years ago. The Bellman-equation of a young individual is:

\[
V_t^Y(B_{b-1,t-1}^Y(i)) = \max \left\{ (1 - \epsilon_t^C) \left[ \frac{1}{1 - \gamma} \left\{ C_{b,t}^{Y,F}(i) \right\}^{1-\gamma} + \frac{\chi}{1 - \gamma} \left\{ C_{b,t}^{Y,I}(i) \right\}^{1-\gamma} \right] + \beta E_t \left( (1 - \omega_t^Y) V_{t+1}^Y(B_{b,t}^Y(i)) + \omega_t^Y V_{t+1}^O(B_{b,t}^Y(i)) \right) \right\}
\]

while the budget constraint is:

\[
(1 + \tau_t^C) C_{b,t}^{Y,F}(i) + p_t^t C_{b,t}^{Y,I}(i) + (1 - \omega_t^Y) B_{b,t}^Y(i) + \omega_t^Y B_{b,t}^O(i) = (1 + r_{t-1}) B_{b-1,t-1}^Y(i) + (1 - \tau_t^{LW}) w_t^L L_{b,t}(i) + w_t^I L_{b,t}(i) + w_t^U U_{b,t}(i) + Profit_{b,t}(i) - T_{b,t}(i)
\]

First-order conditions:

\[
C_{b,t}^{Y,F}(i) : \quad (1 + \epsilon_t^C) \left\{ C_{b,t}^{Y,F}(i) \right\}^{-\gamma} + \lambda_{b,t}^Y (1 + \tau_t^C) = 0
\]

\[
C_{b,t}^{Y,I}(i) : \quad (1 + \epsilon_t^C) \chi \left\{ C_{b,t}^{Y,I}(i) \right\}^{-\gamma} + \lambda_{b,t}^Y p_t^t = 0
\]

\[
B_{b,t}^Y(i) : \quad \beta E_t (1 - \omega_t^Y) V_{b,t}^Y + E_t (1 - \omega_t^Y) \lambda_{b,t}^Y = 0
\]

\[
B_{b,t}^O(i) : \quad \beta E_t \omega_t^Y V_{b,t}^Y + E_t \omega_t^Y \lambda_{b,t}^Y = 0
\]
One-period-ahead Envelope theorem:

\[ E_t V_{Y,t} = -E_t \lambda_{Y_{b+1,t+1}} Y (1 + r_t) \]

Also, from the retired agent’s optimization we know that:

\[ E_t V_{Y,O,t} = -E_t \lambda_{Y_{b+1,t+1}}^{O} (1 + r_t) = -E_t \lambda_{Y_{b+1,t+1}}^{O} (1 + r_t) \]

where \( E_t \lambda_{Y_{b+1,t+1}}^{O} = E_t \lambda_{Y_{b+1,t+1}}^{O} \) because someone who was young in \( t \) retires in \( t + 1 \).

Thus, the Euler equations of the young individual are:

\[ \beta E_t \frac{(1 + \epsilon^C Y_{t+1})(C_{Y,F,b,t}^Y(i))^{\gamma}}{(1 + \epsilon^C Y_{t+1})(C_{b+1,t+1}^Y(i))^{\gamma}}(1 + r_t) \frac{1 + \tau_t^C}{1 + \tau_{t+1}^C} = 1 \]

\[ \beta E_t \frac{(1 + \epsilon^C Y_{t+1})(C_{Y,F,b,t}^Y(i))^{\gamma}}{(1 + \epsilon^C Y_{t+1})(C_{0,t+1}^Y(i))^{\gamma}}(1 + r_t) \frac{1 + \tau_t^C}{1 + \tau_{t+1}^C} = 1 \]

Rearranging:

\[ E_t C_{b+1,t+1}^Y(i) = C_{b,t}^Y(i)(1 + r_t)^{\frac{1}{2}} E_t \Lambda_{t+1} \]

\[ E_t C_{0,t+1}^O(i) = E_t C_{b,t}^Y(i)(1 + r_t)^{\frac{1}{3}} \Lambda_{t+1} \]

Also, we can express each period’s consumption as a function of period-\( t \) consumption and the discount rate:

\[ E_t C_{b+n,t+n}^Y(i) = C_{b,t}^Y(i) E_t \prod_{k=1}^{n} (1 + r_{t+k-1})^{\frac{1}{2}} \Lambda_{t+k} \]

Furthermore, the first-order conditions also imply a substitution between formal and informal goods:

\[ (1 + \epsilon_t^C) \left\{ C_{Y,F,b,t}^Y(i) \right\}^{-\gamma} = (1 + \epsilon_t^C) \chi \left\{ C_{Y,I,b,t}^Y(i) \right\}^{-\gamma} \frac{1 + \tau_t^C}{p_t^F} \]

or with more simple notations:

\[ C_{Y,b,t}^Y(i) = \Upsilon_t C_{b,t}^Y(i) \]

**Individual consumption of a young agent**

Care ought to be taken when deriving the young agent’s individual consumption because old-age incomes and expenditures must be taken into account, too. Moreover, young agents also consider the probability of retirement, for instance, in period \( t \) the probability that a young agent retires in period \( t + 1 \) is \( \omega_t^Y \), while the probability that the same agent retires in period \( t + 2 \) is \( (1 - \omega_t^Y) \omega_t^{Y+1} \). So, the first term of the left-hand
side of this equation shows the stream of lifetime consumption if the agent stays young. From the second term onwards she retires with some probability in each period:

$$
E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega_{t+k-1}^Y) \left[ (1 + \omega_{t+k-1}^T) \left( \prod_{k=1}^{n} \frac{1}{1 + r_{t+k-1}} \right) \right] + \sum_{n=2}^{\infty} \prod_{k=2}^{n} (1 - \omega_{t+k-1}^Y) \left[ (1 + \omega_{t+k-1}^T) \left( \prod_{k=1}^{n} \frac{1}{1 + r_{t+k-1}} \right) \right] + \sum_{n=3}^{\infty} \prod_{k=3}^{n} (1 - \omega_{t+k-1}^Y) \left[ (1 + \omega_{t+k-1}^T) \left( \prod_{k=1}^{n} \frac{1}{1 + r_{t+k-1}} \right) \right] + \ldots
$$

It is easier to express all consumptions in terms of formal goods:

$$
E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega_{t+k-1}^Y) \left[ (1 + \omega_{t+k-1}^T) \left( \prod_{k=1}^{n} \frac{1}{1 + r_{t+k-1}} \right) \right] + \sum_{n=2}^{\infty} \prod_{k=2}^{n} (1 - \omega_{t+k-1}^Y) \left[ (1 + \omega_{t+k-1}^T) \left( \prod_{k=1}^{n} \frac{1}{1 + r_{t+k-1}} \right) \right] + \sum_{n=3}^{\infty} \prod_{k=3}^{n} (1 - \omega_{t+k-1}^Y) \left[ (1 + \omega_{t+k-1}^T) \left( \prod_{k=1}^{n} \frac{1}{1 + r_{t+k-1}} \right) \right] + \ldots
$$

Based on the Euler-equations, we can express expected future consumptions. Let’s consider an agent who is young in period \( t \); her consumption functions in the next periods after retiring are:

$$
E_t C^{O,F}_{n,t+1} (i) = E_t C^{O,F}_{0,t+1} (i) \prod_{k=2}^{n} (1 + r_{t+k-1})^{-\frac{1}{2}} \Lambda_{t+k}
$$

However, if the agent stays young in period \( t + 1 \) and retires after that, then her future old-age consumptions look like:

$$
E_t C^{O,F}_{n,t+2} (i) = E_t C^{O,F}_{0,t+2} (i) \prod_{k=3}^{n} (1 + r_{t+k-1})^{-\frac{1}{2}} \Lambda_{t+k}
$$
Now, we plug them into the intertemporal budget constraint:

\[ E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^y_{t,k-1})(1 + \gamma_c^n)C^{Y,F}_{k,b+n,T+t+n}(0) + P^{t+n}_{k,b+n,T+t+n}(0)T_T^{t+n}) + \]

\[ + E_t \omega^Y (\sum_{n=2}^{\infty} \prod_{k=2}^{n} (1 - \omega^O_{t,k-1})C^{O,F}_{k,t=i}(1 + \gamma_c^n)\frac{1}{2} \Lambda_{t+k}(((1 + \tau^n c^n) + P^{t+n}_{k,b+n,T+t+n})) + \]

\[ + E_t (1 - \omega^Y) \omega^Y (\sum_{n=2}^{\infty} \prod_{k=2}^{n} (1 - \omega^O_{t,k-1})C^{O,F}_{k,t=i}(1 + \gamma_c^n)\frac{1}{2} \Lambda_{t+k}(((1 + \tau^n c^n) + P^{t+n}_{k,b+n,T+t+n})) + \]

\[ = E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^y_{t,k-1}) (1 - \tau^{n} c^{n}_{t} + \omega^Y_{t+n} C^{Y,F}_{k,b+n,T+t+n}(0) + \omega^O_{t+n} C^{O,F}_{k,b+n,T+t+n}(0)) T_T^{t+n}) + \]

\[ + E_t \omega^Y (\sum_{n=2}^{\infty} \prod_{k=2}^{n} (1 - \omega^O_{t,k-1})C^{O,F}_{k,t=i}(1 + \gamma_c^n)\frac{1}{2} \Lambda_{t+k}(((1 + \tau^n c^n) + P^{t+n}_{k,b+n,T+t+n})) + \]

\[ + E_t (1 - \omega^Y) \omega^Y (\sum_{n=2}^{\infty} \prod_{k=2}^{n} (1 - \omega^O_{t,k-1})C^{O,F}_{k,t=i}(1 + \gamma_c^n)\frac{1}{2} \Lambda_{t+k}(((1 + \tau^n c^n) + P^{t+n}_{k,b+n,T+t+n})) + \]

After that, we use the other Euler-equation (the one that shows the substitution between period t and period t + 1 old-age consumption):

\[ E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^y_{t,k-1})(1 + \gamma_c^n)C^{Y,F}_{k,b+n,T+t+n}(0) + P^{t+n}_{k,b+n,T+t+n}(0)T_T^{t+n}) + \]

\[ + E_t \omega^Y (\sum_{n=2}^{\infty} \prod_{k=2}^{n} (1 - \omega^O_{t,k-1})C^{Y,F}_{k,t=i}(1 + \gamma_c^n)\frac{1}{2} \Lambda_{t+k}(((1 + \tau^n c^n) + P^{t+n}_{k,b+n,T+t+n})) + \]

\[ + E_t (1 - \omega^Y) \omega^Y (\sum_{n=2}^{\infty} \prod_{k=2}^{n} (1 - \omega^O_{t,k-1})C^{Y,F}_{k,t=i}(1 + \gamma_c^n)\frac{1}{2} \Lambda_{t+k}(((1 + \tau^n c^n) + P^{t+n}_{k,b+n,T+t+n})) + \]

\[ = E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^y_{t,k-1}) (1 - \tau^{n} c^{n}_{t} + \omega^Y_{t+n} C^{Y,F}_{k,b+n,T+t+n}(0) + \omega^O_{t+n} C^{O,F}_{k,b+n,T+t+n}(0)) T_T^{t+n}) + \]

Concentrating on consumptions:

\[ E_t \sum_{n=0}^{\infty} \prod_{k=1}^{n} (1 - \omega^y_{t,k-1}) C^{Y,F}_{k,b+n,T+t+n}(0)(1 + \tau^{n} c^{n}_{t} + P^{t+n}_{k,b+n,T+t+n}) + \]

\[ + E_t \omega^Y (\sum_{n=2}^{\infty} \prod_{k=2}^{n} (1 - \omega^O_{t,k-1})C^{Y,F}_{k,t=i}(1 + \gamma_c^n)\frac{1}{2} \Lambda_{t+k}(((1 + \tau^n c^n) + P^{t+n}_{k,b+n,T+t+n})) + \]

\[ + E_t (1 - \omega^Y) \omega^Y (\sum_{n=2}^{\infty} \prod_{k=2}^{n} (1 - \omega^O_{t,k-1})C^{Y,F}_{k,t=i}(1 + \gamma_c^n)\frac{1}{2} \Lambda_{t+k}(((1 + \tau^n c^n) + P^{t+n}_{k,b+n,T+t+n})) + \]

+ ...
We rearrange:

\[ C_{b,t}^{Y,F}(i)(1 + r_t^C) + p_t^f \mathcal{T}_t + \left( \frac{\omega^Y_{k+1}(1 + r_t^C) + \frac{4}{5} \Lambda_{t+1}((1 + r_t^C) + p_{t+1}^f \mathcal{T}_{t+1})}{(1 + r_t^C)} \right) + \]

\[ + E_t C_{b+\bar{t},t+1}^{Y,F}(i)(1 - \omega^Y_{t+1})(1 + r_{t+1}^C) + p_{t+1}^f \mathcal{T}_{t+1}) + \]

\[ + E_t C_{b+\bar{t},t+1}^{Y,F}(i)(1 - \omega^Y_{t+1})(1 + r_{t+1}^C) + p_{t+1}^f \mathcal{T}_{t+1}) + \]

Simplifying before recursive substitution:

\[ C_{b,t}^{Y,F}(i)(1 + r_t^C) + p_t^f \mathcal{T}_t + E_t \omega^Y_{t+1}(1 + r_t^C) + \frac{4}{5} \Lambda_{t+1}^O \mathcal{H}_{t+1} + \]

\[ + E_t C_{b+\bar{t},t+1}^{Y,F}(i)(1 - \omega^Y_{t+1})(1 + r_{t+1}^C) + p_{t+1}^f \mathcal{T}_{t+1}) + \]

Now, we can use \( \mathcal{H}_{t+1}^O \) from retired agents’ optimization:

\[ C_{b,t}^{Y,F}(i)(1 + r_t^C) + p_t^f \mathcal{T}_t + E_t \omega^Y_{t+1}(1 + r_t^C) + \frac{4}{5} \Lambda_{t+1}^O \mathcal{H}_{t+1} + \]

\[ + E_t C_{b+\bar{t},t+1}^{Y,F}(i)(1 - \omega^Y_{t+1})(1 + r_{t+1}^C) + p_{t+1}^f \mathcal{T}_{t+1}) + \]

And, using the Euler-equation again (to have period-\( t \) consumption only):

\[ E_t C_{b+\bar{t},t+1}^{Y,F}(i) = C_{b,t}^{Y,F}(i) E_t \prod_{k=1}^{n} (1 + r_t^C) + p_t^f \mathcal{T}_t + \]

Last:

\[ C_{b,t}^{Y,F}(i)(1 + r_t^C) + p_t^f \mathcal{T}_t + E_t \omega^Y_{t+1}(1 + r_t^C) + \frac{4}{5} \Lambda_{t+1}^O \mathcal{H}_{t+1} + \]

\[ + E_t C_{b+\bar{t},t+1}^{Y,F}(i)(1 - \omega^Y_{t+1})(1 + r_{t+1}^C) + p_{t+1}^f \mathcal{T}_{t+1}) + \]

\[ + E_t C_{b+\bar{t},t+1}^{Y,F}(i)(1 - \omega^Y_{t+1})(1 + r_{t+1}^C) + p_{t+1}^f \mathcal{T}_{t+1}) + \]

which is equal to:

\[ C_{b,t}^{Y,F}(i) \mathcal{H}_t^Y \]

where

\[ \mathcal{H}_t^Y = (1 + r_t^C) + p_t^f \mathcal{T}_t + E_t(1 + r_t^C) + \frac{4}{5} \Lambda_{t+1} ((1 - \omega^Y_t) \mathcal{H}_{t+1}^O + \omega^Y_t \mathcal{H}_{t+1}) \]
Similarly to consumption above, the young agent’s budget constraint contains old-age income items, i.e. expected revenue from the pension fund, profits from firms and lump-sum taxes.

\[
x_{b,t}^Y(i) = E_t \omega^Y_t \left( (TR_{0,t+1}^{PG,YO}(i) + TR_{0,t+1}^{FF,YO}(i))\Omega_{t+1}^O + \sum_{n=1}^{\infty} \frac{\Pi_{k=2}^{n-1}(1-\omega_{t+k-1})}{\Pi_{k=2}^{n-1}(1+r_{t+k-1})} (\text{Profit}_{n,t+1}^\text{O} - \tau_{n-1,t+1}(i)) \right) + \\
+ E_t \left(1 - \omega_{t}^Y\right) \omega_{t+1}^Y 1 + r_{t+1} \left((TR_{0,t+2}^{PG,YO}(i) + TR_{0,t+2}^{FF,YO}(i))\Omega_{t+2}^O + \sum_{n=1}^{\infty} \frac{\Pi_{k=2}^{n-1}(1-\omega_{t+k-1})}{\Pi_{k=2}^{n-1}(1+r_{t+k-1})} (\text{Profit}_{n,t+2}^\text{O}) \right) + \\
+ E_t \left(1 - \omega_{t}^Y\right)(1-\omega_{t+1}^Y) \omega_{t+2}^Y (1 + r_{t+2}) \left((TR_{0,t+3}^{PG,YO}(i) + TR_{0,t+3}^{FF,YO}(i))\Omega_{t+3}^O + \sum_{n=1}^{\infty} \frac{\Pi_{k=2}^{n-1}(1-\omega_{t+k-1})}{\Pi_{k=2}^{n-1}(1+r_{t+k-1})} (\text{Profit}_{n,t+3}^\text{O}) \right) + ...
\]

Again, we use that \( TR_{n,t+n}^{PG,YO}(i) = TR_{0,t}^{PG,YO}(i) \forall n > 0 \), and, the same is true for fully funded pensions.

Then, using the definition of \( \mathcal{T}_{O,b,t}(i) \) we can rewrite total old-age income as:

\[
\mathcal{T}_{b,t}^{YO}(i) = E_t \omega_t^Y \left( (TR_{0,t+1}^{PG,YO}(i) + TR_{0,t+1}^{FF,YO}(i))\Omega_{t+1}^O + \mathcal{T}_{O,a,t+1}(i) \right) + \\
+E_t \left(1 - \omega_{t}^Y\right) \omega_{t+1}^Y 1 + r_{t+1} \left((TR_{0,t+2}^{PG,YO}(i) + TR_{0,t+2}^{FF,YO}(i))\Omega_{t+2}^O + \mathcal{T}_{O,t+2}(i) \right) + \\
+E_t \left(1 - \omega_{t}^Y\right)(1-\omega_{t+1}^Y) \omega_{t+2}^Y (1 + r_{t+2}) \left((TR_{0,t+3}^{PG,YO}(i) + TR_{0,t+3}^{FF,YO}(i))\Omega_{t+3}^O + \mathcal{T}_{O,t+3}(i) \right) + ...
\]

which in a recursive way looks like:

\[
\mathcal{T}_{b,t}^{YO}(i) = E_t \omega_t^Y \left( (TR_{0,t+1}^{PG,YO}(i) + TR_{0,t+1}^{FF,YO}(i))\Omega_{t+1}^O + \mathcal{T}_{O,b,t+1}(i) \right) + \\
+E_t \left(1 - \omega_{t}^Y\right) \omega_{t+1}^Y 1 + r_{t+1} \mathcal{T}_{b,t+1}^{YO}(i)
\]

Furthermore, young-age income is (using a new variable \( Inc_t(i) \))

\[
\mathcal{X}_{b,t}^Y(i) = E_t \omega_t^Y \left( (TR_{0,t+1}^{PG,YO}(i) + TR_{0,t+1}^{FF,YO}(i))\Omega_{t+1}^O + \mathcal{T}_{O,b,t+1}(i) \right) + \\
+E_t \left(1 - \omega_{t}^Y\right) \omega_{t+1}^Y \frac{1}{1 + r_{t+1}} \mathcal{T}_{b,t+1}^{YO}(i)
\]

Thus, the individual consumption function of agent \( i \) of cohort \( b \) in period \( t \) is:

\[
\mathcal{H}_{t}^Y C_{b,t}^{Y,F}(i) = \mathcal{T}_{b,t}^Y(i) + \frac{\mathcal{T}_{b,t}^{YO}(i)}{1 + r_{t}} + (1 + r_{t-1})B_{b-1,t-1}(i)
\]

Aggregated consumption of the young cohort
In the first step we need to express the total number of young people. If \( N_{b,t}^Y \) is the number of \( b \)-year old workers, the total number of workers is:

\[
N_t^Y = \sum_{b=0}^{\infty} N_{b,t}^Y
\]

Following the previous idea, we sum up all consumptions, incomes and savings:

\[
\mathcal{H}_t^Y \sum_{b=0}^{\infty} N_{b,t}^Y C_{b,t}^{Y,F}(i) = \sum_{b=0}^{\infty} N_{b,t}^Y I_{b,t}^Y(i) + \frac{1}{1 + r_t} \sum_{b=0}^{\infty} N_{b,t}^Y I_{b,t}^{YO}(i) +
\]

\[
+(1 + r_{t-1}) \sum_{b=0}^{\infty} N_{b,t}^Y B_{b-1,t-1}^Y(i)
\]

where we note that the new young workers in time \( t \) have zero savings from the previous period.

\[
\mathcal{H}_t^Y \sum_{b=0}^{\infty} N_{b,t}^Y C_{b,t}^{Y,F}(i) = \sum_{b=0}^{\infty} N_{b,t}^Y I_{b,t}^Y(i) + \frac{1}{1 + r_t} \sum_{b=0}^{\infty} N_{b,t}^Y I_{b,t}^{YO}(i) +
\]

\[
+(1 + r_{t-1}) \sum_{b=1}^{\infty} N_{b,t}^Y \frac{N_{b-1,t-1}^Y}{N_{b-1,t-1}} B_{b-1,t-1}^Y(i)
\]

Rearranging

\[
\mathcal{H}_t^Y \sum_{b=0}^{\infty} N_{b,t}^Y C_{b,t}^{Y,F}(i) = \sum_{b=0}^{\infty} N_{b,t}^Y I_{b,t}^Y(i) + \frac{1}{1 + r_t} \sum_{b=0}^{\infty} N_{b,t}^Y I_{b,t}^{YO}(i) +
\]

\[
+(1 + r_{t-1})(1 - \omega_{t-1}) \sum_{b=1}^{\infty} N_{b-1,t-1}^Y B_{b-1,t-1}^Y(i)
\]

Aggregate values are defined as:

\[
C_t^{Y,F} \equiv \sum_{b=0}^{\infty} N_{b,t}^Y C_{b,t}^{Y,F}(i)
\]

\[
B_{t-1}^Y \equiv \sum_{b=1}^{\infty} N_{b-1,t-1}^Y B_{b-1,t-1}^Y(i)
\]

\[
I_t^Y \equiv \sum_{b=0}^{\infty} N_{b,t}^Y I_{b,t}^Y(i)
\]

\[
I_t^{YO} \equiv \sum_{b=0}^{\infty} N_{b,t}^Y I_{b,t}^{YO}(i)
\]

It is important to note that in each period, independently from the survival probabilities, each young agent saves for the next period. Hence, the overall savings \( B_{t-1}^Y = \sum_{b=1}^{\infty} N_{b-1,t-1}^Y B_{b-1,t-1}^Y(i) \) is divided among those who remain young and retire.
As a result, the aggregate consumption functions are:

$$\mathcal{H}_t^Y C_t^{Y,F} = T_t^Y + \frac{T_t^{Yo}}{1 + r_t} + (1 + r_{t-1})(1 - \omega_t^Y)B_{t-1}^Y$$

$$C_t^{Y,I} = \Upsilon_t C_t^{Y,F}$$

Now we need to aggregate the supporting variables as well. First of all, we rename individual contemporary income:

$$Inc_{b,t}(i) = (1 - \tau_t^{NW})w_t^F L_{b,t}^F(i) + w_t^L L_{b,t}^L(i) + \omega_t U_{b,t}(i) + Profit_{b,t}(i) - T_{b,t}^Y$$

Aggregating

$$Inc_t = (1 - \tau_t^{NW})w_t^F L_t^F + w_t^L L_t^L + \omega_t U_t + Profit_t - T_t^Y$$

where

$$Profit_t - T_t^Y = \xi (Profit_t - T_t)$$

Aggregating and rearranging

$$\sum_{b=0}^{\infty} N_{b,t}^Y T_{b,t}^Y(i) = \sum_{b=0}^{\infty} N_{b,t}^Y Inc_{b,t}(i) + E_t \frac{(1 - \omega_t^Y)}{1 + r_t} \sum_{b=0}^{\infty} N_{b,t}^Y T_{b+1,t+1}^Y(i)$$

$$= \sum_{b=0}^{\infty} N_{b,t}^Y Inc_{b,t}(i) + E_t \frac{1}{1 + r_t} \sum_{b=0}^{\infty} N_{b+1,t+1}^Y T_{b+1,t+1}^Y(i)$$

Because $T_{t+1}^Y$ contains the income of the new-born people as well, the last term can be rearranged, using the law of large numbers, as:

$$E_t \sum_{b=0}^{\infty} N_{b+1,t+1}^Y T_{b+1,t+1}^Y(i) = E_t T_{t+1}^Y - E_t N_{b,t+1}^Y T_{b,t+1}^Y(i) =$$

$$= E_t T_{t+1}^Y \left( 1 - \frac{N_{b,t+1}^Y}{N_{t+1}^Y} \right) = E_t T_{t+1}^Y \left( 1 - \frac{n_t N_{t}^Y}{N_{t+1}^Y} \right)$$

Then, total young income is:

$$T_t^Y = Inc_t + E_t \frac{1 - \omega_t^Y}{(1 + r_t)(1 + g_t N_{t+1}^Y)} T_{t+1}^Y$$

A similar exercise can be done for pension benefits. First, we define $T_t^{YO}$ which can be rearranged:

$$T_t^{YO} = \sum_{b=0}^{\infty} N_{b,t}^Y T_{b,t}^{YO}(i) = E_t \omega_t^Y \sum_{b=0}^{\infty} N_{b,t}^Y ((TR_{0,t+1}^{F,F,YO}(i) + E_t TR_{0,t+1}^{P,G,YO}(i) O_{t+1} + T_{0,t+1}^O(i))$$

$$+ \frac{(1 - \omega_t^Y)}{(1 + r_{t+1})} \sum_{b=0}^{\infty} N_{b,t}^Y T_{b+1,t+1}^{YO}(i) = E_t N_{0,t+1}^O ((TR_{0,t+1}^{F,F,YO}(i) + TR_{0,t+1}^{P,G,YO}(i) O_{t+1} + T_{0,t+1}^O(i))$$

$$+ E_t \frac{1}{(1 + r_{t+1})} \sum_{b=0}^{\infty} N_{b+1,t+1}^Y T_{b+1,t+1}^{YO}(i)$$
Now, similarly to total young income, the last term can be expressed as:

$$E_t \sum_{b=0}^{\infty} N_{b+1,t+1}^Y T_{b+1,t+1}^{YO}(i) = E_t \frac{1 - \omega^Y_t}{1 + g_{t+1}} T_{t+1}^{YO}$$

Also we know that

$$E_t N_{0,t+1}^O \left( (TR_{0,t+1}^{FG,YO}(i) + TR_{0,t+1}^{PG,YO}(i)) \Omega_{t+1}^O \right) = E_t (TR_{t+1}^{FG,YO} + TR_{t+1}^{PG,YO}) \Omega_{t+1}^O$$

$$E_t N_{0,t+1}^O T_{0,t+1}^O(i) = \frac{\omega^Y_t N_t^Y}{N_{t+1}^Y} T_{t+1}^O = E_t \frac{\omega^Y_t}{(1 + g_{t+1}) s_{t+1}} T_{t+1}^O$$

Finally, the expected income of the young after retiring is

$$T_{t}^{YO} = E_t (TR_{t+1}^{PG,YO} + TR_{t+1}^{FG,YO}) \Omega_{t+1}^O + E_t \left( \frac{\omega^Y_t N_t^Y}{(1 + g_{t+1}) s_{t+1}} T_{t+1}^O \right) + E_t \frac{1 - \omega^Y_t}{(1 + r_{t+1})(1 + g_{t+1})} T_{t+1}^{YO}$$

**Aggregating the young households’ budget constraints**

The individual budget constraint of a young agent is

$$(1 + \tau^C_t) C_{b,t}^{Y,F}(i) + p_t^I C_{b,t}^{Y,I}(i) + (1 - \omega^Y_t) B_{b,t}^Y(i) + \omega^Y_t B_{b,t}^{Y*}(i) = Inc_b(i) + (1 + r_{t-1}) B_{b-1,t-1}^Y$$

Aggregating

$$\sum_{b=0}^{\infty} N_{b,t}^Y ((1 + \tau^C_t) C_{b,t}^{Y,F}(i) + p_t^I C_{b,t}^{Y,I}(i)) + \sum_{b=0}^{\infty} N_{b,t}^Y (1 - \omega^Y_t) B_{b,t}^Y(i) + \sum_{b=0}^{\infty} N_{b,t}^Y \omega^Y_t B_{b,t}^{Y*}(i) =$$

$$= \sum_{b=1}^{\infty} N_{b,t}^Y Inc_b(i) + (1 + r_{t-1}) \sum_{b=1}^{\infty} N_{b,t}^Y B_{b-1,t-1}^Y(i)$$

where the definition of aggregate savings is:

$$\sum_{b=1}^{\infty} N_{b,t}^Y B_{b-1,t-1}^Y(i) = \sum_{b=1}^{\infty} (1 - \omega^Y_{t-1}) N_{b-1,t-1}^Y B_{b-1,t-1}^Y$$

After aggregation, there is no difference between the $B_t^Y$ and $B_t^{Y*}$. So, we can easily express the aggregate budget constraint:

$$(1 + \tau^C_t) C_t^{Y,F} + p_t^I C_t^{Y,I} + B_t^Y = Inc_t + (1 + r_{t-1})(1 - \omega^Y_{t-1}) B_{t-1}^Y$$

**B The public old-age pension systems**

**B.1 Pay-as-you-go pension system**

In a PAYG regime, all public revenues finance all public expenditures (also pension benefits):

$$Rev_t = \tau^C_t C^F_t + \tau^L_t w^F_t L^F_t + T_t$$

$$\tau^L_t = \tau^P_t + (1 - \Xi)(\tau^{SCW}_t + \tau^{SCF}_t)$$

$$Exp_t = Gou_t + w^I_t U_t + TR_t^{PG}$$

where $\Xi$ is 0 in a PAYG regime, 1 in a fully funded regime, but it can be time-variant as well.
The number of the just-retired agents (those who were young one period before) is

\[ N_{0,t}^O = \sum_{b=1}^{\infty} \omega_{t-1}^Y N_{b-1,t-1}^Y \]

Individual \(i\)'s pension in the year of retirement \(t\) is based on replacement rate \(\nu_t\) and the average of the last \(Y\) years' income:

\[ TR_{0,t}^{PG,YO}(i) = \nu_t IB_{b-1,t}^Y(i) \]

where

\[ IB_{b-1,t}^Y(i) = \frac{1}{Y} w_{t-1} F_{b-1,t-1}(i) + \frac{Y - 1}{Y} IB_{b-2,t-1}^Y(i) \]

Aggregating the last expression implies

\[ IB_t^Y = \sum_{b=1}^{\infty} N_{b-1,t-1}^Y IB_{b-1,t}^Y(i) = \]

\[ \frac{1}{Y} w_{t-1} F_{t-1} + \frac{Y - 1}{Y} \sum_{b=2}^{\infty} N_{b-1,t-1}^Y \frac{N_{b-2,t-2}^Y}{N_{b-2,t-2}} IB_{b-2,t-1}^Y(i) \]

It is also true that

\[ IB_t^Y = \frac{1}{Y} w_{t-1} F_{t-1} + \frac{Y - 1}{Y} (1 - \omega_{t-2}^Y) \sum_{b=2}^{\infty} N_{b-2,t-2}^Y IB_{b-2,t-1}(i) \]

which can be rearranged

\[ IB_t^Y = \frac{1}{Y} w_{t-1} F_{t-1} + \frac{Y - 1}{Y} (1 - \omega_{t-2}^Y) IB_{t-1}(i) \]

Then

\[ IB_t^Y = \frac{1}{Y} w_{t-1} F_{t-1} + \frac{Y - 1}{Y} (1 - \omega_{t-2}^Y) IB_{t-1}(i) \]

We need this to aggregate the just-retired pension benefits

\[ N_{0,t}^O TR_{0,t}^{PG,YO}(i) = \nu_t N_{0,t}^O IB_{b-1,t}^Y(i) = \nu_t \omega_{t-1}^Y \sum_{b=1}^{\infty} N_{b-1,t-1}^Y IB_{b-1,t}^Y(i) \]

Total pension expenditure of the just retired is simple with the new notations

\[ TR_t^{PG,YO} = \nu_t \omega_{t-1}^Y IB_t^Y \]

Furthermore, the total pension expenditure of all retired people is

\[ TR_t^P = TR_t^{PG,YO} + (1 - \omega_{t-1}^O) TR_{t-1}^{PG,YO} + (1 - \omega_{t-2}^O) TR_{t-2}^{PG,YO} + ... \]

which can be rewritten as

\[ TR_t^P = TR_t^{PG,YO} + (1 - \omega_{t-1}^O) TR_{t-1}^P \]
Fully funded pension system

In the fully funded regime, pension benefits are not financed from the public budget, but are based on individual savings:

\[ E_xpt = Gov_t + w^U_t U_t \]

Agent \( i \) has a private account, where she can accumulate her own pension wealth:

\[ B^Y_{b,t}(i) = \Xi(\tau_t^{SSCW} + \tau_t^{SSCF})w^F_t L^F_{b,t}(i) + (1 + r_{t-1})B^Y_{b-1,t-1}(i) \]

Aggregating

\[ B^Y_t = \sum_{b=0}^{\infty} N^Y_{b,t}B^Y_{b,t}(i) = \Xi(\tau_t^{SSCW} + \tau_t^{SSCF})w^F_t \sum_{b=0}^{\infty} N^Y_{b,t}L^F_{b,t}(i) + (1 + r_{t-1})\sum_{b=1}^{\infty} N^Y_{b,t}B^Y_{b-1,t-1}(i) \]

Then the last term can be rearranged small

\[ \sum_{b=1}^{\infty} N^Y_{b,t}B^Y_{b-1,t-1}(i) = (1 - \omega^Y_{t-1})\sum_{b=1}^{\infty} N^Y_{b-1,t-1}B^Y_{b-1,t-1}(i) = (1 - \omega^Y_{t-1})B^Y_{t-1} \]

Hence

\[ B^Y_t = \Xi(\tau_t^{SSCW} + \tau_t^{SSCF})w^F_t L^F_t + (1 + r_{t-1})(1 - \omega^Y_{t-1})B^Y_{t-1} \]

At the moment of retirement, the just-retired’s benefits are calculated based on pension wealth that has been accumulated thus far, projected life expectancy and discount rates. In each period the pension fund transfers this amount, while the rest remains on the account.

\[ (1 + r_{t-1})B^Y_{b-1,t-1}(i) = TR^{FF,YO}_{0,t}(i) + E_t \frac{1 - \omega^O_t}{1 + r_t} TR^{FF,YO}_{1,t+1}(i) + \]

\[ + E_t \frac{1 - \omega^O_t(1 - \omega^O_{t+1})}{(1 + r_t)(1 + r_{t+1})} TR^{FF,YO}_{2,t+2}(i) + ... = TR^{FF,YO}_{0,t}(i)\Omega^O_t \]

Aggregating and rearranging

\[ (1 + r_{t-1})N^Y_{0,t}B^Y_{b-1,t-1}(i) = N^O_{0,t}TR^{FF,YO}_{0,t}(i)\Omega^O_t = TR^{FF,YO}_{0,t}\Omega^O_t \]

\[ (1 + r_{t-1})N^Y_{0,t}B^Y_{b-1,t-1}(i) = TR^{FF,YO}_{0,t}\Omega^O_t \]

\[ (1 + r_{t-1})\omega^Y_{t-1}B^Y_{t-1}(i) = TR^{FF,YO}_{t}\Omega^O_t \]

Overall total pension expenditure in the fully funded regime can be given in a recursive way, like that of the PAYG regime, as

\[ TR^{FF}_t = TR^{FF,YO}_t + (1 - \omega^O_{t-1})TR^{FF}_{t-1} \]
Finally, we can express savings in each period after pension benefits were deducted

\[(1 + r_{t-1})B_{a-1,t-1}^{O,*}(i) = TR_{a,t}^{FF,YO}(i) + (1 - \omega_{t}^{O})B_{a,t}^{O,*}(i)\]

Aggregating and rearranging

\[(1 + r_{t-1}) \sum_{a=0}^{\infty} N_{a,t}^{O} B_{a-1,t-1}^{O,*}(i) = \sum_{a=0}^{\infty} N_{a,t}^{O} TR_{a,t}^{FF,YO}(i) + (1 - \omega_{t-1}^{O}) \sum_{a=0}^{\infty} N_{a,t}^{O} B_{a,t}^{O,*}(i) = TR_{t}^{FF} + B_{t}^{O,*}\]

The left-hand side of the above equation can be rearranged as

\[\sum_{a=0}^{\infty} N_{a,t}^{O} B_{a-1,t-1}^{O,*}(i) = N_{0,t}^{O} B_{-1,t-1}^{O,*}(i) + \sum_{a=1}^{\infty} N_{a,t}^{O} B_{a-1,t-1}^{O,*}(i) = N_{0,t}^{O} B_{-1,t-1}^{O,*}(i) + (1 - \omega_{t-1}^{O}) \sum_{a=1}^{\infty} N_{a-1,t-1}^{O} B_{a-1,t-1}^{O,*}(i)\]

Last, we use that \(B_{a-1,t-1}^{O,*}(i) = B_{b-1,t-1}^{Y, last,*}(i)\) because the initial old pension wealth is equal to pension savings accumulated over the lifetime. Also, we rewrite \(N_{a,t}^{O}\). So the first term of the above equation becomes

\[\omega_{t-1}^{Y} \sum_{b=1}^{\infty} N_{b-1,t-1}^{Y} B_{b-1,t-1}^{Y, last,*}(i) + (1 - \omega_{t-1}^{O}) \sum_{a=1}^{\infty} N_{a-1,t-1}^{O} B_{a-1,t-1}^{O,*}(i)\]

Finally the market clearing equations for old-age pension savings are

\[(1 + r_{t-1}) \omega_{t-1}^{Y} B_{t-1}^{Y,*} + (1 + r_{t-1}) B_{t-1}^{O,*} = TR_{t}^{FF} + B_{t}^{O,*}\]

\[B_{t}^{*} = B_{t}^{Y,*} + B_{t}^{O,*}\]

### C Normalized equations

Each variable must be detrended; individual variables are normalized with technology \((A_t)\) and aggregate variables are normalized with technology and population \((N_t)\), because technology and population growth are included in the model. This section lists all the final equations of the model; the detrended variable \(x_t\) is denoted by \(\ddot{x}_t\).

**Demography:**

\[s_t = \frac{(1 - \omega_{t-1}^{O})}{(1 - \omega_{t-1}^{Y} + n_t)} s_{t-1} + \frac{\omega_{t-1}^{Y}}{(1 - \omega_{t-1}^{Y} + n_t)}\]

\[s_t^Y = \frac{1}{1 + s_t}\]

\[1 + g_t^{N,Y} = 1 - \omega_{t-1}^{Y} + n_t\]

\[1 + g_t^{N,O} = (1 - \omega_{t-1}^{O}) + \frac{\omega_{t-1}}{s_{t-1}}\]

\[1 + g_t^{N} = (1 + g_t^{N,Y}) \frac{1 + s_t}{1 + s_{t-1}}\]

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Overlapping generations:

\[ \mathcal{H}_t^O c_t^{O,F} = (T \hat{R}_t^{PG} + T \hat{R}_t^{FF})\Omega_t^O + \tilde{T}_t^O + \frac{(1 + r_{t-1})}{1 + g_t} \left( \omega_t^Y B_{t-1}^Y + B_{t-1}^\Omega \right) \]

\[ \tilde{T}_t^O = (1 - \xi) \left( P_r \tilde{v}_t - \tilde{T}_t \right) + E_t \frac{1 + g_{t+1}}{1 + r_t} \frac{1 - \omega_t^O}{1 + g_{t+1}^O} \tilde{T}^O_{t+1} \]

\[ \mathcal{H}_t^O = (1 + r_t^C) + p_t^I \Upsilon_t + E_t(1 - \omega_t^O)(1 + r_t)^{\frac{1}{2} - 1} \Lambda_{t+1} \mathcal{H}_{t+1}^O \]

\[ \mathcal{H}_t^Y C_t^{Y,F} = \tilde{T}_t^Y + \frac{\tilde{T}^Y_{t+1}}{1 + r_t} + \frac{(1 + r_{t-1})(1 - \omega_{t-1})}{1 + g_t} B_{t-1}^Y \]

\[ \Omega_t^O = 1 + E_t \frac{1 - \omega_t^O}{1 + r_t} \Omega_{t+1}^O \]

\[ C_t^{O,I} = \Upsilon_t C_t^{O,F} \]

\[ E_t \lambda_{t+1} = E_t \left\{ \beta \frac{1 + \epsilon_{t+1}^C}{1 + \epsilon_t^C} \frac{1 + r_{t+1}}{1 + r_t} \right\} \]

\[ \Upsilon_t = \left\{ \frac{1 + r_t^C}{p_t^I} \right\} \]

\[ I_t \tilde{c}_t = (1 - \tau_t) \tilde{w}_t^F L_t^F + \tilde{w}_t^I L_t^I + \tilde{w}_t^Y U_t + \xi \left( P_r \tilde{v}_t - \tilde{T}_t \right) \]

\[ \tilde{T}_t^Y = I_t \tilde{c}_t + E_t \frac{1 - \omega_t^Y}{1 + r_t} \frac{1 + g_{t+1}}{1 + g_{t+1}^N} \tilde{T}^O_{t+1} \]

\[ \tilde{T}^Y_{t+1} = E_t (1 + g_{t+1}) \left( (1 + r_{t+1}) \left( (1 + g_{t+1}^N) \tilde{T}^O_{t+1} \right) + \frac{1}{1 + g_{t+1}^N} \right) \]

\[ + E_t \frac{(1 - \omega_{t+1}^Y)(1 + g_{t+1})}{1 + r_{t+1}} \frac{1 - \omega_{t+1}^Y}{1 + g_{t+1}^N} \tilde{T}^O_{t+1} \]

\[ C_t^{Y,I} = \Upsilon_t C_t^{Y,F} \]

\[ (1 + r_t^C) C_t^{Y,F} + p_t^I C_t^{Y,I} + B_t^Y = I_t \tilde{c}_t + \frac{(1 + r_{t-1})}{1 + g_t} (1 - \omega_{t-1}^Y) B_{t-1}^Y \]

Formal firms - except labor market:

\[ R \left( \frac{1 + \pi_t^F}{(1 + \pi_{t-1})^\gamma} \right) = \frac{\phi}{2} \left( \frac{1 + \pi_t^F}{(1 + \pi_{t-1})^\gamma} - 1 \right)^2 \]

\[ 1 + \frac{1}{\varphi - 1} R \left( \frac{1 + \pi_t^F}{(1 + \pi_{t-1})^\gamma} \right) + \frac{1}{\varphi - 1} R' \left( \frac{1 + \pi_t^F}{(1 + \pi_{t-1})^\gamma} \right) + \frac{1 + \pi_t^F}{(1 + \pi_{t-1})^\gamma} \]

\[ - E_t \frac{1}{\varphi - 1} \left( 1 + g_{t+1} \right) \frac{Y_{t+1}^F}{\gamma} \frac{R' \left( \frac{1 + \pi_t^F}{(1 + \pi_{t-1})^\gamma} \right) \left( \frac{1 + \pi_t^F}{(1 + \pi_{t-1})^\gamma} \right) - \frac{1 + \pi_t^F}{(1 + \pi_{t-1})^\gamma}}{1 + r_t} = \frac{\varphi}{\varphi - 1} m_c^F \]

\[ m_c^F = E_t \left( \frac{\pi_t^{K,F}}{\alpha^F} \right) \left[ \frac{(1 + r_t^{SSCF}) \tilde{w}_t^F + h_c^F - h_c^{F+1}(1 - \phi^F,F)}{1 + g_{t+1}} \right] \frac{(1 - \alpha^F)}{A_t^F(1 - \alpha^F)} \]

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\[
\frac{1}{1 + g_t} K_{t-1}^F = \alpha^F mc_t^F \frac{r_t^F Y_t^F}{L_t^F}
\]

\[
mc_t^F (1 - \alpha^F) \frac{Y_t^F}{L_t^F} - (1 + \tau_{t}^{SSCF}) w_t^F = h_{t}^F - E_t \frac{h_{t+1}^F(1 - pr_t^F,F)}{1 + r_t}(1 + g_{t+1}^A)
\]

\[
\text{profit}_t^F = Y_t^F - \tau_t^{K,F} K_{t-1}^F \frac{1}{1 + g_t} - (1 + \tau_t^{SSCW}) w_t^F L_t^F - R(\cdot) Y_t^F - h_{t}^F H_t^F
\]

\[
E_t(r_{t+1}^K + Q_{t+1}^F (1 - \delta)) = Q_t^F (1 + r_t)
\]

\[
1 = Q_t^F \left(1 - S \left(\frac{I_{t+1}^F}{I_{t-1}^F}\right) - S' \left(\frac{I_{t+1}^F}{I_{t-1}^F}\right) \left(\frac{I_{t+1}^F}{I_{t-1}^F}\right)^2\right)
\]

\[
1 + \frac{1}{\varphi - 1} \frac{R}{1 + g_{t+1}} Y_{t+1}^F \frac{R'}{(1 + \pi_{t+1}^F) (1 + \pi_{t-1}^F) - \varphi - 1} = \frac{1 + \frac{\pi_{t+1}^F}{\pi_{t-1}^F} (1 + \pi_{t+1}^F) (1 + \pi_{t-1}^F)}{1 + \frac{\pi_{t+1}^F}{\pi_{t-1}^F}}}
\]

Informal firms - except labor market:

\[
R \left(\frac{1 + \pi_t^F}{1 + \pi_{t-1}^F}\right) = \frac{\phi_p}{2} \left(\frac{1 + \pi_t^F}{1 + \pi_{t-1}^F}\right) - 1
\]

\[
mc_t^I \left(\frac{r_t^K}{\alpha^I}\right) \left(\frac{1 + \frac{\pi_t^I}{\pi_{t-1}^I}}{\alpha^I}\right) \left(\frac{w_t^I + h_{t}^I - \frac{h_{t+1}^I (1 - pr_t^F,I)}{1 + r_t}}{A_t^I(1 - \alpha^I)}\right)
\]

\[
\frac{1}{1 + g_t} K_{t-1}^I = \alpha^I mc_t^I \frac{r_t^I Y_t^I}{L_t^I}
\]

\[
mc_t^I (1 - \alpha^I) \frac{Y_t^I}{L_t^I} - \frac{w_t^I}{L_t^I} = h_{t}^I - E_t \frac{h_{t+1}^I (1 - pr_t^F,I)}{1 + r_t}(1 + g_{t+1}^A)
\]

\[
\text{profit}_t^I = p_t^I Y_t^I - \tau_t^{K,I} K_{t-1}^I \frac{1}{1 + g_t} - \frac{w_t^I L_t^I}{1 + g_t} - R(\cdot) Y_t^I - h_{t}^I H_t^I
\]

\[
E_t(r_{t+1}^K + Q_{t+1}^I (1 - \delta)) = Q_t^I (1 + r_t)
\]

\[
Q_t^I \left(1 - S \left(\frac{I_{t+1}^I}{I_{t-1}^I}\right) - S' \left(\frac{I_{t+1}^I}{I_{t-1}^I}\right) \left(\frac{I_{t+1}^I}{I_{t-1}^I}\right)^2\right)
\]

\[
1 + \frac{1}{1 + r_t} Q_{t+1}^I S' \left(\frac{I_{t+1}^I}{I_{t-1}^I}\right) \left(\frac{I_{t+1}^I}{I_{t-1}^I}\right)^2 = 1
\]

\[
\text{Informal firms - except labor market:}
\]

\[
\frac{1}{1 + g_t} K_{t-1}^F = \alpha^F mc_t^F \frac{r_t^F Y_t^F}{L_t^F}
\]
Labor market with wage bargaining:

\[ h^F \]_i = \kappa^F (pr^H>F)^{\alpha_H^C} \\
\]

\[ pt^H>F_i = \frac{\tilde{H}^F_i}{U^t_{i-1} + g^t_i + pr^F,F L^F_i + pr^F,I L^F_i} + \tilde{H}^F_i \]

\[ \tilde{L}^F_i = (1 - pr^F) \frac{L^F_{i-1}}{1 + g^t_i} + \tilde{H}^F_i \]

\[ \frac{\sigma_F}{1 - \sigma_F} \tilde{h}^c_i - \frac{1 - \tau^{LW}}{1 + \tau^{SSCF}} = (1 - \tau^{LW}) w^F_i - \tilde{w}^U_i + \]

\[ \frac{1 + g^{A}_{i+1}}{1 + r^t_i} \left[ (1 - pr^F,F_i) \left( \frac{\sigma_F}{1 - \sigma_F} \tilde{h}^c_{i+1} \frac{1 - \tau^{LW}}{1 + \tau^{SSCF}} \right) \right] - \]

\[ \frac{1 + g^{A}_{i+1}}{1 + r^t_i} \left[ (1 - pr^F,F_i) \right. \left( \frac{\sigma_I}{1 - \sigma_I} \tilde{h}^c_{i+1} \frac{1 - \tau^{LW}}{1 + \tau^{SSCF}} \right) \]

Government\(^{26}\)

\[ R^{\tilde{C}}v_i = \tau^{C} \tilde{G}^F_i + \tau^{L} \tilde{w}^F_i \tilde{L}^F_i + \tilde{T}_i \]

\[ \tau^{LW} = \tau^{PL}_i + \tau^{SSCF}_i \]

\[ \tilde{B}_i = \frac{1}{1 + g_t} B_{i-1} - \tilde{G} B_t \]

\[ PB_t = R^{\tilde{C}}v_t - E^{\tilde{C}} \tilde{p}_t \]

\[ GB_t = PB_t - r_{i-1} \frac{1}{1 + g_t} \tilde{B}_i \]

\[ \tilde{T}_t = \eta T + (1 - \eta) \left[ \rho \tilde{T}_{i-1} + (1 - \rho) (GB^{\tilde{target}}_t - \tilde{G} B_i) \right] \]

PAYG pension system\(^{27}\)

\[ \tau^{P} = \tau^{PL} + (1 - \Xi) (\tau^{SSCW} + \tau^{SSCF}) \]

\[ E^{\tilde{C}} \tilde{p}_t = \tilde{G} \tilde{v}_t + \tilde{w}^U_i \tilde{U}_i + TR^{\tilde{PG}}_t \]

\(^{26}\)There is a similar rule for all other fiscal instruments, not only for lump-sum taxes.

\(^{27}\)IB\(^{H} \) in the third and fourth equations is normalized by \( A_{i-1} N_{i-1} \).
\[ I\ddot{B}_t^Y = \frac{1}{Y} w_{t-1}^F L_{t-1}^F + \frac{Y - 1 - \omega_{t-2}^Y}{Y} IB_{t-1}^Y \]

\[ TR_{t}^{\tilde{F},G,YO} = \nu_t \frac{\omega_{t-1}^Y}{1 + g_t} \tilde{B}_{t-1}^Y \]

\[ TR_{t}^{\tilde{F},PG} = \frac{(1 - \omega_{t-1}^O)}{1 + g_t} TR_{t-1}^{\tilde{F},PG} \]

Fully funded pension system:

\[ E\ddot{x}_t = G\ddot{v}_t + w_t^U \ddot{U}_t \]

\[ B_t^{\tilde{Y}*} = \Xi (\tau_t^{SSC} + \tau_t^{SCF}) w_t^F L_t^F + \frac{(1 + r_{t-1})}{1 + g_t} (1 - \omega_{t-1}^Y) B_{t-1}^{\tilde{Y}*} \]

\[ \frac{(1 + r_{t-1})}{1 + g_t} \omega_{t-1}^Y B_{t-1}^{\tilde{Y}*} = TR_{t}^{\tilde{F},F,YO} \Omega_t^O \]

\[ TR_{t}^{\tilde{F},FF} = TR_{t}^{\tilde{F},F,YO} \left( 1 - \omega_{t-1}^F \right) \frac{1}{1 + g_t} TR_{t-1}^{\tilde{F},FF} \]

\[ \frac{(1 + r_{t-1})}{1 + g_t} \omega_{t-1}^Y B_{t-1}^{\tilde{Y}*} + \frac{(1 + r_{t-1})}{1 + g_t} B_{t-1}^{\tilde{O}*} = TR_{t}^{\tilde{F},FF} + B_{t}^{\tilde{O}*} \]

\[ B_t^* = B_t^{\tilde{Y}*} + B_t^{\tilde{O}*} \]

Monetary policy:

\[ 1 + i_t = (1 + i_{t-1})^\rho_t E_t \left( (1 + r) (1 + \pi_{t+1}^F)^{\delta_x} \right)^{1-\rho_t} e^t \]

\[ 1 + i_t = (1 + r_t) E_t (1 + \pi_{t+1}^F) \]

Market clearing:

\[ \ddot{U}_t = s_t^Y - \ddot{L}_t^F - \ddot{L}_t^I \]

\[ Profit_t = Profit_t^F + Profit_t^I \]

\[ B_t + Q_t^F K_t^F + Q_t^I K_t^I = B_t^{\tilde{Y}} + B_t^{\tilde{O}} + \ddot{B}_t^* \]

\[ Y_t^F = \ddot{C}_t^F + \dddot{I} \dddot{v}_t + G\dddot{v}_t + h c_t^F H_t^F + R \left( \frac{P_t^F}{P_{t-1}^F} \right) + \dddot{I} \dddot{v}_t^I S \left( \frac{I_{\dddot{v}_t}^F}{I_{\dddot{v}_t}^{I-1}} \right) + \]

\[ + I_{\dddot{v}_t}^I S \left( \frac{I_{\dddot{v}_t}^I}{I_{\dddot{v}_t}^{I-1}} \right) \]

\[ \dddot{Y}_t^I = \dddot{C}_t^I + h c_t^I H_t^I + R \left( \frac{P_t^I}{P_{t-1}^I} \right) \]

\[ \dddot{C}_t^F = \dddot{C}_t^{\tilde{Y},F} + \dddot{C}_t^{\tilde{O},F} \]

\[ \dddot{C}_t^I = \dddot{C}_t^{\tilde{Y},I} + \dddot{C}_t^{\tilde{O},I} \]

\[ I_{\dddot{v}_t} = I_{\dddot{v}_t}^F + I_{\dddot{v}_t}^I \]

\[ G\dddot{D}_P_t = \dddot{C}_t + I_{\dddot{v}_t} + G\dddot{v}_t \]

\[ \dddot{C}_t = \dddot{C}_t^F + p_t^F \dddot{C}_t^I \]

\[ \dddot{Y}_t = Y_t^F + p_t^I \dddot{Y}_t^I \]
D Steady state of the model

The steady state is solved numerically. First, we specify initial guesses for the following variables: \( r, p^I, \frac{Y^F}{Y}, pr^{H,F} \) and \( pr^{H,I} \). Then, as a function of initial guesses, we can determine the variables of production, labor market and bargaining and those of the government and pension systems. Finally, we turn to the consumption and savings functions. At the end, using the market clearing equations and some other leftover equations, we can check whether our initial guesses are correct. First, we calculate each variable in terms of production, then, after calculating \( \tilde{Y} \) itself, we can find the steady-state levels of the variables. Now, we describe the steps we use to calculate the model’s steady state in detail.

First, the demographic equations are:

\[
\begin{align*}
s &= \frac{\omega Y}{(1 - \omega Y + n)} \\
1 + g^{N,Y} &= 1 - \omega Y + n \\
1 + g^{N,O} &= (1 - \omega O) + \frac{\omega Y}{s} \\
1 + g^{N} &= (1 + g^{N,Y})\frac{1 + s}{1 + s}
\end{align*}
\]

Additionally, the balanced growth trend can be given by:

\[
(1 + g) = (1 + g^A)(1 + g^N)
\]

where \( g^A \) is the exogenous productivity growth.

Then, we need to guess an initial value for \( r, p^I, \frac{\tilde{Y}^I}{Y}, pr^{H,F} \) and \( pr^{H,I} \) which are verified by the Newton-Raphson algorithm. Assuming \( \pi^F = 0 \) in the steady state implies:

\[
i = r
\]

In the steady state the quadratic adjustment costs are zero, so:

\[
Q^F = 1 \\
Q^I = 1
\]

and from the no-arbitrage conditions:

\[
r^{K,F} = r + \delta \\
r^{K,I} = r + \delta
\]

Then, we can calculate the marginal costs from the Phillips-curves:

\[
mc^F = \frac{\varphi - 1}{\varphi} \\
mc^I = \frac{\varphi - 1}{\varphi} p^I
\]
Because we have a guess for $\tilde{Y}_I$ and $p^I$, formal production share is given by:

$$\frac{\tilde{Y}_F}{Y} = 1 - p^I \frac{\tilde{Y}_I}{Y}$$

Also, we can express the capital-output ratios from the capital demands:

$$\frac{\tilde{K}_F}{Y} = (1 + g) \alpha_F \frac{mc F \tilde{Y}_F}{rK,F}$$
$$\frac{\tilde{K}_I}{Y} = (1 + g) \alpha_I \frac{mc I \tilde{Y}_I}{rK,I}$$

and then the investment-output ratios are:

$$\frac{I_{\tilde{v}_F}}{Y} = \frac{\tilde{K}_F}{Y} \left( 1 - (1 - \delta) \frac{1}{1 + g} \right)$$
$$\frac{I_{\tilde{v}_I}}{Y} = \frac{\tilde{K}_I}{Y} \left( 1 - (1 - \delta) \frac{1}{1 + g} \right)$$
$$\frac{I_{\tilde{v}}}{Y} = \frac{I_{\tilde{v}_F}}{Y} + \frac{I_{\tilde{v}_I}}{Y}$$

As a next step, from the marginal cost functions we calculate the gross wages (assuming that $A_F = 1$ and $A_I = 1$):

$$(1 + \tau^{SSCF}) \tilde{w}^F = (1 - \alpha^F) \left[ \frac{mc F}{rK,F} \alpha_F \right]^{\frac{1}{1 - \alpha^F}}$$
$$\tilde{w}^I = (1 - \alpha^I) \left[ \frac{mc I}{rK,I} \alpha_I \right]^{\frac{1}{1 - \alpha^I}}$$

Now, we know the ratios of hiring costs to wages ($Wage\_Ratio^F$ and $Wage\_Ratio^I$), so we can endogenously determine $\kappa_F$ and $\kappa_I$:

$$hc^F = Wage\_Ratio^F (1 + \tau^{SSCF}) \tilde{w}^F$$
$$hc^I = Wage\_Ratio^I \tilde{w}^I$$
$$\kappa^F = \frac{hc^F}{prH,F}$$
$$\kappa^I = \frac{hc^I}{prH,I}$$

Then, using the labor demand equations we can calculate the labor-output ratios:

$$\frac{\tilde{L}_F}{Y} = \frac{\tilde{Y}_F}{Y} \frac{mc F (1 - \alpha^F)}{(1 + \tau^{SSCF}) \tilde{w}^F + hc^F - \frac{hc^F (1 - prH,F)}{1 + r}(1 + g A)}$$
$$\frac{\tilde{L}_I}{Y} = \frac{\tilde{Y}_I}{Y} \frac{mc I (1 - \alpha^I)}{\tilde{w}^I + hc^I - \frac{hc^I (1 - prH,I)}{1 + r}(1 + g A)}$$
Now, the hiring-output ratios are:

\[
\frac{\tilde{H}^F}{Y} = \frac{\tilde{L}^F}{Y} \left( 1 - (1 - pr^F) \frac{\tilde{L}^F}{1 + g^N} \right)
\]

\[
\frac{\tilde{H}^I}{Y} = \frac{\tilde{L}^I}{Y} \left( 1 - (1 - pr^{F,I}) \frac{\tilde{L}^I}{1 + g^N} \right)
\]

And, the unemployment-output ratio can be given by using the \(pr^{H,F}\) equation:

\[
\frac{\tilde{U}}{Y} = (1 + g^N) \frac{\tilde{H}^F}{Y} - \frac{pr^{H,F} \tilde{L}^F}{Y} - \frac{pr^{F,I} \tilde{L}^I}{Y}
\]

The remaining three labor market equations (\(pr^{H,F}\) and the two wage bargaining equations) are used by the Newton-Raphson algorithm to verify the initial guesses for \(\tilde{Y}^I/Y\), \(pr^{H,F}\) and \(pr^{H,I}\).

Next, we can calculate profits in total production:

\[
\frac{\mbox{profit}^F}{Y} = p^F \frac{\tilde{Y}^F}{Y} - r^{K,F} \frac{\tilde{K}^F}{Y} \frac{1}{1 + g} - (1 + r^{SSCW}) \frac{\tilde{w}^F \tilde{L}^F}{Y} - \frac{hc^F \tilde{H}^F}{Y}
\]

\[
\frac{\mbox{profit}^I}{Y} = p^I \frac{\tilde{Y}^I}{Y} - r^{K,I} \frac{\tilde{K}^I}{Y} \frac{1}{1 + g} - \frac{\tilde{w}^I \tilde{L}^I}{Y} - \frac{hc^I \tilde{H}^I}{Y}
\]

Using the goods market clearing conditions, we can express formal and informal consumption as a share of total production:

\[
\frac{\tilde{C}^F}{Y} = \frac{\tilde{Y}^F}{Y} - \frac{\tilde{I}^{nv}}{Y} - \frac{\tilde{Gov}}{Y} - \frac{hc^F \tilde{H}^F}{Y}
\]

\[
\frac{\tilde{C}^I}{Y} = \frac{\tilde{Y}^I}{Y} - \frac{hc^I \tilde{H}^I}{Y}
\]

As a result, total consumption-output ratio is:

\[
\frac{\tilde{C}}{Y} = \frac{\tilde{C}^F}{Y} + p^I \frac{\tilde{C}^I}{Y}
\]

Because all variables are thus far expressed as a share of production, we need the GDP-production ratio:

\[
\frac{\tilde{GDP}}{Y} = \frac{\tilde{C}}{Y} + \frac{\tilde{I}^{nv}}{Y} + \frac{\tilde{Gov}}{Y}
\]

We observe all distortionary tax revenues as a share of GDP; hence, we can calculate the effective tax rates, which are consistent with the model’s labor and goods markets.
Now we can express all other fiscal variables (the initial calibration is done with a PAYG pension plan):

\[ \hat{IB}^Y = \frac{1}{Y} \frac{\hat{w}_F^L \hat{L}_F^Y}{Y} \]

\[ TR_{PG,YO}^Y = \nu \frac{\omega^Y \hat{IB}^Y}{1 + g} \]

\[ TR_{PG}^Y = \frac{TR_{PG,YO}^Y}{1 - \left(1 - \frac{\omega^O}{1 + g}\right)} \]

\[ E^{xp} \frac{\hat{Y}}{Y} = \frac{Gov}{Y} + \hat{w}_I \hat{U} + \frac{\hat{TR}_{PG}^Y}{Y} \]

Taking into account the data on the government debt to GDP ratio, we calculate the lump-sum tax to GDP ratio:

\[ \hat{GB}^Y = \left(\frac{1}{1 + g} - 1\right) \hat{B}^Y \]

\[ \hat{P}_B^Y = GB^Y \frac{1}{Y} + \frac{1}{1 + g} \hat{B}^Y \]

\[ \hat{Rev}^Y = \frac{\hat{P}_B^Y}{Y} + \frac{\hat{Exp}^Y}{Y} \]

\[ \tilde{T}^Y = \frac{\hat{Rev}^Y}{Y} - \tau^C \hat{C}_F^Y - \tau^L \hat{w}_F \hat{L}_F^Y \]

Now we express all supporting variables of the households:

\[ \tilde{T}^O = \frac{(1 - \xi) \left(\frac{\hat{Profit}^Y}{Y} - \frac{\hat{T}}{Y}\right)}{1 - \frac{1 + g - 1}{1 + r} \frac{1 - \omega^O}{1 + g}} \]

\[ \Omega^O = \frac{1}{1 - \frac{1 - \omega^O}{1 + r}} \]

\[ \Lambda = \beta^\frac{1}{\gamma} \]

\[ \Upsilon = \left\{ \chi \frac{1 + \tau^C}{p^I} \right\}^{\frac{1}{\gamma}} \]

\[ \mathcal{H}^O = \frac{(1 + \tau^C) + p^I \Upsilon}{1 - (1 - \omega^O)(1 + r)^{\frac{1}{\gamma} - 1} \Lambda} \]

\[ \mathcal{H}^Y = \frac{(1 + \tau^C) + p^I \Upsilon + (1 + r)^{\frac{1}{\gamma} - 1} \Lambda \omega^Y \mathcal{H}^O}{1 - (1 + r)^{\frac{1}{\gamma} - 1} \Lambda(1 - \omega^Y)} \]

\[ \frac{\hat{nc}}{Y} = \frac{(1 - \tau^{LW}) \hat{w}_F \hat{L}_F^Y}{Y} + \frac{\hat{w}_I \hat{L}_I^Y + \hat{w}_U \hat{U}}{Y} + \xi \left( \frac{\hat{Profit}^Y}{Y} - \frac{\hat{T}}{Y} \right) \]

\[ \tilde{T}^Y = \frac{\hat{nc}}{Y} - \frac{(1 - \omega^Y)(1 + g)}{(1 + r)(1 + g) \gamma} \]

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\[
\frac{\tilde{I}Y_o}{Y} = (1 + g) \left( \frac{(TRP_{G,Y} + TRF_{E,Y}) \Omega Y + \omega Y}{(1 + g^{n\tau})} \cdot \frac{\tilde{I}O}{Y} \right) \\
\]

Using the young agents’ consumption function, budget constraint and the remaining first-order conditions, we can express the savings to total production ratio:

\[
\mathcal{H}^Y \frac{C_{Y,F} Y}{Y} = \frac{\tilde{I}Y}{Y} + \frac{\tilde{I}Y_o}{Y} \frac{(1 + r)(1 - \omega Y) \tilde{B}Y}{1 + g} \frac{1 + \tau C}{1 + \tau C} + p^f Y \\
\]

where we plug in the informal consumption to production ratio, in order to express the formal consumption to production ratio:

\[
\frac{C_{Y,F} Y}{Y} = \frac{\tilde{I}Y}{Y} + \frac{(1 + r)(1 - \omega Y) - 1}{1 + \tau C} \frac{\tilde{B}Y}{Y} \\
\]

Then, plugging this back in the consumption function:

\[
\mathcal{H}^Y \left( \frac{\tilde{I}nc}{Y} + \frac{(1 + r)(1 - \omega Y) - 1}{1 + \tau C} \frac{\tilde{B}Y}{Y} \right) = \frac{\tilde{I}Y}{Y} + \frac{\tilde{I}Y_o}{Y} \frac{(1 + r)(1 - \omega Y) \tilde{B}Y}{1 + g} \frac{1 + \tau C}{1 + \tau C} + p^f Y \\
\]

So, the steady-state savings as a share of production are:

\[
\tilde{B}Y = \frac{(1 + \tau C) + p^f Y}{\mathcal{H}^Y} \left( \frac{\tilde{I}Y}{Y} + \frac{\tilde{I}Y_o}{Y} \frac{1 + \tau C}{1 + \tau C} \right) - \frac{\tilde{I}nc}{Y} \\
\]

As a next step we can express the steady-state formal and informal consumptions:

\[
\frac{C_{Y,F} Y}{Y} = \frac{\tilde{I}Y}{Y} + \frac{\tilde{I}Y_o}{Y} \frac{1 + \tau C}{1 + \tau C} + \frac{(1 + r)(1 - \omega Y) \tilde{B}Y}{Y} \\
\]

and, using the formal goods market clearing condition, the old households’ formal and informal consumption functions are:

\[
\frac{C_{O,F} Y}{Y} = \frac{\tilde{C}_F}{Y} - \frac{C_{Y,F} Y}{Y} \\
\]

\[
\frac{C_{O,I} Y}{Y} = \gamma \frac{C_{O,F} Y}{Y} \\
\]

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Finally, the retired savings-to-production ratio looks like:

\[
\frac{\bar{B}^O}{\bar{Y}} = H^O \frac{C_{\bar{Y},F}}{Y} - \left( \frac{T_{\bar{R}PC}}{Y} + \frac{T_{\bar{R}FF}}{Y} \right) \Omega^O - \frac{T_{\bar{C}}}{Y} \frac{1}{1+g} \frac{\omega Y \bar{B}^Y}{Y}
\]

In the last step we calculate total production based on the unemployment equation:

\[
\hat{U} = \frac{\hat{Y}}{\hat{Y} - \frac{\hat{L}^F}{\hat{Y}} - \frac{\hat{L}^I}{\hat{Y}}}
\]

\
\hat{Y} = \frac{\hat{U}}{\hat{Y} + \frac{\hat{L}^F}{\hat{Y}} + \frac{\hat{L}^I}{\hat{Y}}}
\

Now, the levels of all variables can be finally expressed. The five remaining equations (see below) are used to verify the initial guesses for \( r, p^I, \frac{Y^I}{Y}, p^{H,F} \) and \( p^{H,I} \). If the guesses are correct, the left- and right-hand sides of the remaining equations are equal. Otherwise the Newton-Raphson method chooses new initial values for these five variables; this process goes on until no new initial values must be chosen.

\[
res_1 = \frac{\sigma_F}{1 - \sigma_F} \hat{h}^F \frac{1 - \tau_{LW}}{1 + \tau_{SSCF}} - \left\{ (1 - \tau_{LW}) \hat{w}^F - \hat{w}^U + \frac{1 + g^A}{1 + r} \left[ (1 - pr^{F,F})(1 - pr^{H,F}) \left( \frac{\sigma_F}{1 - \sigma_F} \hat{h}^F \frac{1 - \tau_{LW}}{1 + \tau_{SSCF}} \right) \right] - \right.
\]

\[
\left. - \frac{1 + g^A}{1 + r} \left[ (1 - pr^{F,F})pr^{H,I} \left( \frac{\sigma_I}{1 - \sigma_I} \hat{h}^I \right) \right] \right\}
\]

\[
res_2 = pr^{H,I} - \left( \frac{\hat{H}^I}{\hat{U} + \frac{\hat{L}^F}{1 + g^\omega} + \frac{\hat{L}^I}{1 + g^\omega}} \right)
\]

\[
res_3 = \frac{\sigma_I}{1 - \sigma_I} \hat{h}^I - \left\{ \hat{w}^I - \hat{w}^U + \frac{1 + g^A}{1 + r} \left[ (1 - pr^{F,I}) \left( \frac{\sigma_I}{1 - \sigma_I} \hat{h}^I \right) \right] - \right.
\]

\[
\left. - \frac{1 + g^A}{1 + r} \left[ pr^{H,F}(1 - pr^{F,I}) \frac{\sigma_F}{1 - \sigma_F} \hat{h}^F \frac{1 - \tau_{LW}}{1 + \tau_{SSCF}} \right] \right\}
\]

\[
res_4 = \bar{B} + Q^F \tilde{K}^F + Q^I \tilde{K}^I - \left( \tilde{B}^Y + \tilde{B}^O \right)
\]

\[
res_5 = \tilde{C}^I - \widetilde{C}^{Y,I} C^{\tilde{O}}
\]
References


