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Contents

Abstract

In this paper we provide a brief introduction to the literature on agent-based financial modelling and, more specifically, artificial stock market modelling. In the selective literature review two broad categories of artificial stock market models are discussed: models based on hard-wired rules and models with learning and systemic adaptation. The paper discusses pros and cons of agent-based financial modelling as opposed to the standard representative-agent approach. We advocate the need for the proper account of market complexity, agent heterogeneity, bounded rationality and adaptive (though not simplistic) expectations in financial modelling. We also argue that intelligent adaptation in highly uncertain environment is key to understanding actual financial market behaviour and we resort to a specific area of artificial intelligence theory, namely reinforcement learning, as one plausible and economically appealing algorithm of adaptation and learning.

Keywords: Agent-based financial modelling, artificial stock market, complex dynamical system, market efficiency agent heterogeneity, reinforcement learning.
JEL classification: G10, G11, G14, Y20.

Santrauka

Šiame straipsnyje trumpai pristatoma agentų elgsena pagrįsto finansinio modeliavimo literatūra, didžiausią dėmesį skiriant dirbtinės finansų rinkos modeliams. Atrankioje apžvalgoje pateikiami modeliai suskirstyti į dvi bendras kategorijas: tai – modeliai, kuriuose agentų elsena pagrįsta aiškiai nustatytomis taisyklėmis, bei modeliai, pasižymintys agentų mokymusi ir sistemos lygmens adaptacija. Straipsnyje išreiškiama nuostata, kad kuriant finansinius modelius reikia tinkamai atsižvelgti į rinkos kompleksiškumą, agentų heterogeniškumą, ribotą racionalumą ir adaptyvius (tačiau ne pernelyg primityvius) lūkesčius. Čia taip pat argumentuojama, kad protinga adaptacija dideliu neabrėžtumu pasižyminčioje aplinkoje yra esminis modeliavimo aspektas, kuris padėtų suvokti svarbiausius finansų rinkų funkcionavimo principus. Tuo tikslu šiame pradiniame susijusių tyrimų etape trumpai idėjiniu lygmeniu nagrinėjama konkreti dirbtinės intelekto teorijos sritis – paskatinamasis mokymasis, kuris iš principo galėtų pasiūlyti ekonomiškai priimtinus mokymosi ir sisteminės adaptacijos algoritmus.

1. Introduction

In 2008 the world economy encountered a global financial crisis, caused by a mix of a global asset price bubble, overwhelming irrational exuberance and systemic mistakes of economic agents and financial market participants. Against this background, the standard financial theory, based on the efficient market hypothesis and rational representative agent paradigm, seems to be losing touch with reality. Unfortunately, there are no satisfactory alternatives yet, but with growing computing power, modelling possibilities expand and new promising frontiers of research emerge. One of them could be agent-based finance. Agent-based financial models take account of fundamental features undoubtedly inherent to the real world financial markets, such as agent heterogeneity, bounded rationality and complex interaction of agents. More generally, these computer models give researchers great flexibility to model interesting features of real world phenomena. This will eventually allow very realistic modelling of financial markets, goods markets or the economy as a whole. Before this vision materialises, however, a major breakthrough in realistic modelling of intelligent, but boundedly rational human behaviour is needed. It can only be achieved by blending and expanding advancements of the economic theory, cognitive psychology and artificial intelligence theory.

Mainstream capital market theories widely regarded financial markets as extremely efficient price determination mechanisms. Moreover, trading processes themselves were effectively excluded from the analysis by applying strong market clearing assumptions. Now one can observe the gradual paradigm shift in the financial literature, as financial markets are being increasingly viewed as complex dynamical systems, consisting of interacting atomistic agents whose complex interaction and private learning result in some systemic adaptation but not necessarily high market-level efficiency. The ongoing paradigm shift is by no means a merely academic debate. It is of great importance for market participants and policy makers. It is pretty obvious that in recent years economists' blind belief in nearly perfect markets' self-regulation abilities outshined the premonition of the looming global financial catastrophe and arguably even paved the way for it. Hence, the current crisis offers economic researchers a good reason to devote more effort for enhancing generative understanding of market processes rather than concentrate on equilibrium relations.

The current paper is aimed at providing a non-specialist reader with a very brief and selective introduction to the emerging area of agent-based financial modelling and, more specifically, the artificial stock market (ASM) modelling. To our knowledge, such analysis has not been conducted in the Lithuanian economic literature. An intuitive discussion of problems of the standard financial modelling as well as specific modelling alternatives will hopefully stimulate academic discussion among Lithuanian economists and be policy-relevant.

The paper is organised as follows. In Section 2 we give a general discussion about agent-based financial models as an alternative and a complement to standard financial theories. A brief ASM literature review and presentation of some specific ASM design issues are provided in Section 3. Section 4 is devoted for intuitive presentation of basic principles of the standard reinforcement learning and its relevance in the context of financial modelling. Section 5 contains concluding remarks.

2. Agent-based financial models versus the standard representative-agent paradigm

Much of the mainstream financial theory builds on the efficient market hypothesis (EMH) and the rational representative agent paradigm. These presumptions have clearly played a crucial role in shaping the widely accepted understanding of risk, determinants of asset prices, portfolio management principles, etc. Yet there is a growing need to recognise that the gap between this idealisation and reality may be too substantial for the theory to grasp correctly the essence of functioning of financial markets, as the standard theory arguably abstracts from the salient features of examined phenomena. The central question is whether the financial market can be seen as a perfect (or near-perfect) price determination mechanism. There are a lot of theoretical caveats and empirical anomalies associated with standard financial theories, strong assumptions of perfect rationality and the efficient market hypothesis.

2.1. General discussion of homogeneity, perfect rationality and market efficiency

Theoretically, the homogeneous agent assumption is far from innocuous, as at the heart of finance lies the collective discovery of securities prices in the process of trading, i.e. as a result of the interaction of heterogeneous agents. It is a plain fact, which hardly requires any scientific inquiry, that investors have a bewildering variety of views, expectations and preferences. They possess different amounts of information, and their financial decisions vary greatly in the level of sophistication. In reality, the financial market awakes to life and trading takes place exactly owing to this heterogeneity of market participants. An assumption that each individual, or the aggregate market behaviour, can be approximated by some average or a fictitious representative agent inevitably leads to the loss of a large degree of freedom. By assuming this, one clearly risks attributing effects of changes in individual agents' perceptions and strategies to something like the representative agent's consumption smoothing preferences.

Even stronger is the perfect rationality assumption. It is obvious that, as already noted by Simon (1957) half a century ago, individuals act in a highly uncertain environment and their natural computing abilities are limited, while information search and analysis are costly and time consuming. All of this implies that even if perfect rationality was feasible in the information collection, processing and decision making sense, it would simply be too costly economically. Paradoxically, it is hardly rational to attempt being perfectly rational. Moreover, a large thread of literature of psychology and behavioural finance instigated by laboratory experiments of Kahneman and Tversky (1973) and Tversky and Kahneman (1974) suggest that economic behaviour is often better explained by simple heuristic rules and irrational biases rather than by dynamic optimisation.

The conflict between the perfect rationality idea and common sense deepens further if we consider specifically financial markets. This is related to an inherently large impact of expectations on the aggregate financial market behaviour. For example, consider the standard line of thinking that individuals invest in risky assets so as to optimise their consumption patterns. Owing to their different preferences, under the heterogeneity assumption they all have potentially different valuations of expected fundamental payoffs (e.g. an expected stream of stock dividends). As investors want to optimally adjust their investment positions, aggregate supply and demand shift, and market prices of risky assets change as a result. The magnitude of this change is largely unpredictable because in reality there is no way of knowing idiosyncratic factors affecting each investor's asset supply and demand curves. In the short run, stock prices are arguably more affected by these idiosyncratic shocks than by relatively infrequent news on the structural changes of the processes materially affecting fundamentals – this idea can be traced back to Keynes (1936); see also Cutler et al. (1989).

Probably even more importantly, every market participant has some marginal impact on these price fluctuations and their expectations about likely price changes may become self-fulfilling. Any signal, such as good news related to a specific stock, may lead to investors' coincident actions. This often triggers changes in the stock price in a predictable direction, which implies that immediately following the news it can become optimal for short-term speculators to buy the stock irrespective of actual fundamentals. There is nothing to prevent under- or over-reactions to the news, hence it is highly unrealistic to assume that the actual market price always coincides with some fundamental value. Partially self-fulfilling expectations may lead to multiple sunspot equilibria, which, of course, are not consistent with rational expectations by definition. In other words, Muth's (1961) rational expectations hypothesis can simply be seen as an elegant way to exclude "ad hoc" forecasting rules and market psychology from economic modelling (Hommes, 2006) but due to the self-referential nature of predictions they may be deductively indeterminate. In reality market participants are more likely to form expectations inductively (Arthur, 1995) – subjective expectations are formed, tested and changed dynamically, as market conditions change and market participants gain experience or interpret (possibly erroneously) plentiful information signals.

Every reasonable person knows from their own experience that there is no conceivable mechanism ensuring that their economic or social behaviour is perfectly rational. Yet proponents of the EMH hypothesis argue that perfect rationality can be an emergent feature of the financial market[1]. There are claims, for instance, that the existence of arbitrage traders, evolutionary competition and generally offsetting each other noise traders' bets may ensure that securities prices always reflect fundamentals correctly. The idea (which is also known in the literature as the Friedman hypothesis) that poor performance drives non-rational investors out of the market is indeed appealing. However, investment in stocks and some other securities is not a zero-sum game. Stock prices generate positive returns in the long run in the overwhelming majority of cases. Hence, it is not clear why non-rational investors, especially passive investors, should "die out" – they may well enjoy decent returns to their less-than-rational (e.g. passive) investment strategies. Moreover, their army is constantly replenished with new inexperienced and hence non-rational investors. The argument of a negligible impact of emotional and non-rational traders can also be challenged. It is exactly them, rather than "fundamentalist" traders, who are more likely to react to non-fundamental headline news and push market prices in the predictable direction, acting as a powerful

[1] Emergent features are systemic features that cannot be deduced by simply scaling individual behaviour – see Chen and Yeh (2002) for discussion.

market-moving force and thereby imposing “rules of the game”. Moreover, it is well known that sophisticated traders, instead of acting as a stabilising force, may try to exploit the resultant predictable market movements. For instance, Frankel and Froot (1987) conclude from their survey that investors often recognise a considerable price deviation from their perceived fundamentals but nevertheless they find it logical to follow the trend until it reaches some turning point.

There are also a number of empirical problems with traditional financial models based on perfect rationality and EMH assumptions. Financial literature discusses quite a few empirical anomalies, i.e. empirical regularities that are not explained by the theory. Probably the most famous one is the equity premium puzzle raised by Mehra and Prescott (1985). Stock returns (or equity risk premia) seem to be too high to be explained by investors’ consumption optimisation behaviour, implying implausibly high levels of their risk aversion. Shiller (1981) and others have noted that stock prices exhibit excessive volatility, as compared to changes in fundamentals. There are also some indications that markets may be more predictable than the EMH hypothesis suggests (Campbell and Shiller, 1988, Lo and MacKinlay, 1988). Empirical facts, such as large trading volumes, fat tails of returns distribution or persistent stock price volatility, are not well understood either (LeBaron, 2006). And, of course, booms, busts and financial crises – which are the absolutely salient features of today’s economic reality and should be placed really high on economists’ research agenda – are in a discord with the standard rational representative agent paradigm and the EMH hypothesis.

2.2. The new paradigm – markets as complex agent-based systems

Making strong assumptions in standard financial models may have been about the only way to make theoretical generalisations about the market behaviour. But now that the growing computing power and advancing computational methods have enabled researchers to relax some of those assumptions, economics and finance are witnessing an important paradigm shift towards a behavioural, agent-based approach. According to this approach, markets are seen as complex dynamical systems consisting of heterogeneous learning, boundedly rational heterogeneous agents (see Hommes, 2006, LeBaron, 2006).

Computational study of these dynamical systems of interacting agents is what agent-based computational finance is all about. Let us very briefly discuss the principal attributes of the research object of agent-based financial models. Naturally, at the centre-stage are agents. Agents, in this context, are given quite a broad meaning. According to Tesfatsion (2006), they comprise bundled data and behavioural methods representing an entity in a computationally constructed environment. They can range from active, learning and data-gathering decision-makers (e.g. investors, consumers, workers), their social groupings (e.g. firms, banks, families) and institutions (e.g. markets, regulatory systems) to passive world features such as infrastructure. From the operational point of view, they are similar to objects and object groups in the object-oriented programming, whereas agent-based models technically are collections of algorithms embodied in those entities termed “agents”. The possibility to develop composite and hierarchical structures of computational agents implies that they can become arbitrarily complex and may greatly surpass their analytical counterparts of standard models in respect of reflecting salient features of the real world entities.

The interdisciplinary nature of the notion of an agent also leads one to the realm of the computer science. Here, an autonomous agent is understood as a system situated in, and part of, an environment, which senses that environment, and acts on it, over time, in pursuit of its own agenda (Franklin and Graesser, 1997). If agents are capable of learning to achieve their goals more efficiently or their population as a whole continuously adapts to be better suited to survive, the artificial intelligence theory comes into play. Learning and adaptation are crucially important in agent-based modelling since the ultimate goal of any economic analysis is to model the actual *human intelligent* behaviour and its consequences at the individual or aggregate level.

Agents form complex adaptive systems. A system is said to be complex if it is constituted of interacting elements (agents) and exhibits emergent properties, i.e. properties inherent to the system but not necessarily to individual agents. Depending on the complexity of studied phenomena, complex adaptive systems may include reactive agents (capable of reacting in a systematic way to changing environmental conditions), goal-directed agents (reactive and capable of directing some of their actions to achieving their goals) and planning agents (goal-directed and capable of exerting some control over environment). It is important that these systems are self-sufficient or dynamically complete, i.e. they may evolve – without interventions from the modeller – in reaction to exogenous environmental changes or even as a result of merely endogenous interaction of agents. For a more thorough discussion of basic elements and principles of agent-based computational economics and finance see Tesfatsion (2006).

Once agents are put together in a complex system, systemic patterns resulting from agents' interaction may be observed and the system's reaction to exogenous shocks can be analysed. Hence, agent-based financial models basically are a simulation tool. This determines the delicate position of agent-based financial modelling among standard scientific inference methods: deduction-based theoretical models (i.e. theoretical generalisation from certain assumptions) and induction-based empirical models (recognition of systematic patterns in the empirical data). Simulation, and agent-based modelling in particular, does not allow one to prove theoretical propositions, nor does it directly measure real world phenomena, so there is always a risk of analysing an artificial world too remote from reality. On the other hand, simulation, just like deductive analysis, is based on explicit assumptions, which in many cases are much more realistic than in analytical models. If those assumptions and parameters of exogenous processes are calibrated to match empirical data, then simulation analysis does lend itself to drawing valuable inductive inferences about the real world behaviour. Generally, as Axelrod and Tesfatsion (2006) observe, simulation permits increased understanding of systems through controlled computational experiments. Epstein (2006) also notes the importance of agent-based modelling as a tool for generative explanation. While most economic and financial theory deals with analysis of equilibria, he argues that it is not enough to claim that a system – be it an economy, financial market or other social grouping consisting of rational agents – if put in the Nash equilibrium, stays there. For a fuller understanding of system's behaviour for generativists it is important to understand how the local autonomous interactions of atomistic, heterogeneous and boundedly rational agents generate the observed macro-level regularities and how the system reaches, if reaches at all, the equilibrium. Moreover, plausibility of any equilibrium patterns at the macro-level is required by generativists to be confirmed by generating it from suitable microspecifications. As Epstein (1999) puts it, "If you didn't grow it, you didn't explain it". In general, agent-based economic and financial modelling has several primary objectives – the abovementioned empirical understanding of macro-level regularities, normative understanding of potential institutional and policy improvements, and qualitative insight and theory generation through examination of simulated behaviour (see Tesfatsion, 2006, for extended discussion).

Key features of agent-based financial models are well summarised by Epstein (2006). The most important feature and actually the primary reason for departing from standard analytical settings is heterogeneity of agents. Agents may differ in their preferences, skills, decision rules, information sets, levels of wealth, etc., and their characteristics may change over time independently of others. Agent behaviour is generally characterised by bounded rationality, which arises both from limited information and limited computational capacities of agents. Agent interactions are autonomous, i.e. there is no central planner, Walrasian auctioneer or other central controllers, though interaction rules, behavioural norms and institutional settings may be arbitrarily sophisticated. Agent-based models also require an explicit agent interaction network, which may be centralised or decentralised (in which case agents interact locally), and their financial decisions may be influenced by information flows through social networks. Finally, analysis of non-equilibrium dynamics of analysed systems in agent-based modelling is of no lesser importance than studying equilibrium properties.

Agent-based modelling clearly gives researchers a large degree of much desired flexibility necessary to understand the real world financial market phenomena. Unfortunately, this poses problems too. Much room for manoeuvre implies that agent-based models vary to such an extent that they lack some unifying fundament, which could help to develop this interesting area of research into a solid theory with an established methodology and conventional wisdom about basic building blocks. There are also serious difficulties related to modelling micro-level behaviour. Having made the pretty obvious proposition that the real world investors are less than fully rational, researchers face difficult conceptual issues related to deciding what then governs agents' behaviour is and how to model it. Should modelled micro-behaviour match that of human subjects in laboratory experiments? Should modellers deliberately include in their agent-based models behavioural biases and heuristic rules confirmed by experimental data? Should artificial agents apply decision rules that have some substantiation in the theoretical representative agent models? How learning and expectation formation processes should be modelled? Answers to these questions, of course, depend on the problem at hand but, as stressed by Duffy (2006), in principle they are not yet systemically addressed by researchers of this field. Dealing with these issues, Duffy suggests that external validation of simulation results should not be limited to comparison of aggregate outcomes of simulated and real world phenomena. He advocates careful selection of model parameters based on experiments with human subjects and suggests seeking stronger external validation by comparing simulation results to those obtained in the experiments with humans, both at the micro- and macro-level.

3. Brief ASM literature review and important ASM design issues

The literature on the development of the artificial stock markets[2] (ASM) is one specific area of the agent-based modelling. In this section we discuss some important issues arising when designing ASMs and review – though only briefly and selectively – the literature on ASMs.

3.1. ASM building principles and problems

All ASM modellers have to deal with challenging design issues and face modelling tradeoffs. These include, but are not limited to, the choice of agents' preferences and objectives, properties of securities, mechanisms of price determination, expectations formation, evolution and learning algorithms, timing issues and benchmarks.

Since stock markets usually constitute extremely complex environment, one immediate problem is that any degree of realism imposes huge computational costs and results in the loss of analytical tractability. The problem is most severe in modelling agents' intelligent behaviour. It is further aggravated by the fact that the decision making processes are unobservable and there is too little theoretical guidance on how to model these processes realistically. For these reasons ASM models usually are highly stylised, and in many cases model settings are kept close to certain benchmarks – standard theoretical rational expectations models or tractable and well understood special cases of agent-based models. This generally strengthens the credibility of ASM models as tools of generative explanation of systemic equilibria derived under strong assumptions and eases interpretation of simulation results.

Agents and their decision-making processes occupy the centre-stage in agent-based models of stock markets and are the main source of diversity of ASMs. The agent design might vary from budget constrained zero-intelligence agents to sophisticated artificially intelligent decision-making entities. It is worth noting in passing that some agent designs lack dynamic integrity and lasting identity inherent to human subjects, which brings interpretation of these artificial agents closer to competing bundles of strategies rather than to actual investors. Agents usually are given utility functions, and utility levels associated with different strategies are important in driving agents' behaviour or determining their "fitness" in the evolutionary selection process. Agents may derive utility from different sources, e.g., consumption, wealth or returns. A serious limitation but a very natural one, given the complexity of the model environment, is that in most cases agents are myopic in that they care only about one-period utility and do not attempt to carry out dynamic optimisation.

In simplest settings agents' behaviour may be completely random (constrained only by budget constraints to make it economically interesting, as in Gode and Sunder, 1993). Alternatively, they may follow strict decision rules or choose conditional strategies from a (dynamically evolving) bundle of strategies. These rules may be suggested by standard theories or may mimic popular actual investment strategies. The central design question in the ASM models based on artificial intelligence is how

[2] ASMs are also referred in the literature to as simulated stock markets and agent-based models of stock markets.

agents choose investment strategies and how the pool of available strategies evolves. It should be noted that in almost all models, agents – taken individually – have very limited intelligence, whereas systemic adaptation and strategy improvement mostly take place at the population level. Most ASM models employ Holland's (1975) genetic algorithm technique to drive the evolution of strategies. In such algorithms, inspired by the theory of biological evolution, strategy pools evolve as a result of the rule crossover, mutation and evolutionary survival of the fittest rules. Alternatively, agents may choose their investment strategies or form their forecasts based on neural network or simple econometric forecasting techniques. Another learning possibility is the Roth and Erev (1995) type stimulus-response learning[3] (discussed, e.g., in Brenner, 2006, or Duffy, 2006), which leans on the simple idea that actions yielding larger payoffs tend to be repeated more frequently. More technically rigorous and economically appealing is the reinforcement learning mechanism established in the artificial intelligence literature (see, e.g., Sutton and Barto, 1998). This approach has been hardly ever used in the context of ASM modelling, partly due to some known problems of application of such algorithms in multi-agent settings. But, in our view, the inability to ensure that the learned strategies are asymptotically optimal should not preclude modellers from taking advantage of these economically intuitive learning algorithms.

Specification of the market setting is another very important ASM design question. ASM modellers usually simplify the portfolio allocation task, and in most models there are only two types of securities traded – a risky dividend-paying stock and a riskless bond. Moreover, pricing in the bond market is typically shut down by assuming constant interest rates. Pricing of the stock is hence determined by both fundamental factors and interaction of heterogeneous agents (and dynamics of their expectations), though some features of these determinants may be greatly simplified for analytical or computational purposes. For instance, the dividend process may not be modelled specifically, or dividend can be assumed to be paid out every trading period, which is a highly unrealistic but quite necessary assumption in the myopic agent environment. Next critical issue in specifying the market setting, is the choice of the price determination mechanism. According to LeBaron (2001, 2006), there are four major classes of price determination mechanisms: (*i*) gradual price adjustment, in which case individual sell and buy orders (for a given price) are aggregated and in the next trading period the price is gradually shifted by the market-maker in response to excessive supply or demand (the market is almost never in equilibrium), (*ii*) immediate market clearing, whereby the market clearing price is computed from agents' supply and demand functions (the market is always in temporary equilibrium), (*iii*) randomly matched trading, whereby trade takes place between randomly matched agent pairs, and (*iv*) an order book, which most closely models the actual trading process on order-driven automated stock exchanges. In this context it is useful to note the problem of trade synchronicity – something, which is not an issue in standard analytical representative agent models where there is simply no trade. Actually, the real world traders arrive in the market and make their orders asynchronously, which may lead to strategic intra-period interaction. There are some attempts to build event-driven ASMs instead of

[3] It is also known in the agent-based modelling literature as the reinforcement learning but it should not be confused with the formal reinforcement learning algorithm developed in the artificial intelligence literature.

ASMs evolving in equal time increments. However, owing to technical and conceptual difficulties, most ASM models assume that trading decisions are taken by all agents simultaneously without having any strategic interaction of this type.

3.2. Brief ASM literature review

Now let us turn to some specific models. The area of agent-based modelling of stock markets has been active for about two decades. The need for this alternative modelling of stock market behaviour arose from dissatisfaction with the abovementioned strong assumptions of the standard financial theory, its neglect for simple investment behaviour rules that are often used by finance practitioners and inability of standard models to explain satisfactorily the real world stock price dynamics (e.g. the US stock market crash on 17th October 1987 and, of course, the ongoing global financial meltdown). We divide (somewhat arbitrarily) the models into two broad categories – (*i*) models based on stochastic, heuristic and standard theory-implied behavioural rules and (*ii*) models with learning agents or evolutionary systemic adaptation. The latter category is arguably more promising and interesting.

3.2.1. ASM models based on random, heuristic and hard-wired behavioural rules

The first group of ASM models generally investigate whether interaction of heterogeneous agents, who base their decisions on simple deterministic rules or even act in a random manner, might induce stock price movements qualitatively similar to those observed in real stock markets. Agents in these models usually follow simple, "hard-wired" investment rules. In order to generate interesting market dynamics without sacrificing model parsimony and tractability, it is very common to allow just a few investment strategies (these are the so-called few-type models; see LeBaron, 2006). For instance, market participants may be broadly categorised as "fundamentalists", "chartists" and "noise traders". Fundamentalists base their investment decisions on fundamental information about stock dividend potential, chartists rely on technical analysis of time series of stock prices, whereas noise traders may base their investment decisions on erroneous signals about fundamentals, follow aggregate market behaviour or, say, simply behave in a random manner. Though such strategies bear some resemblance to the real world investment behaviour, the problem lies in determining the distribution of different investor types in model population, as this distribution may play a key role in shaping the aggregate market behaviour. Clearly, market developments are influenced by relative popularity of different strategies. Under different circumstances some strategies may become dominant and optimal to follow, hence in some models agents are allowed to switch to different strategies. They can switch to alternative strategies on their performance, or underperforming investors may simply exert smaller influence on market developments due to their smaller financial wealth.

The origins of this strand of literature are linked to the few-type models of foreign exchange markets proposed by Frankel and Froot (1988), Kirman (1991), De Grauwe et al. (1993) and others. A prominent early example of the few-type model of a stock market is Kim and Markowitz (1989). In their stylised model there are two types of agents that pursue either the portfolio rebalancing strategy or the portfolio insurance strategy. Rebalancers aim at keeping a constant fraction of their assets in a risky stock, while portfolio insurers try to ensure the minimum level of wealth by defensively selling some of the stock holdings if the minimum threshold approaches. The rebalancing strategy works as a market stabilising force (a stock price decline spurs stock buying), whereas the portfolio insurance strategy amplifies market fluctuations (a stock price decline prompts stock selling). With simple expectation formation and decision rules and with the uncertainty induced by monetary shocks, the model shows that some investment strategies, namely the abovementioned portfolio insurance strategy, may have a sizeable destabilising effect on the market and can be partly responsible for market crashes. Later models developed along this line of research are much more detail-rich but they are still aimed at giving a plausible explanation of complicated empirical market dynamics, which is not quite consistent with standard financial models. In detailed models of Lux (1995), Lux and Marchesi (1999, 2000) much of market dynamics is attributed to agents' endogenous switching to different trading strategies depending on the prevailing majority opinion. Another popular idea in the ASMs is that some "smart money" traders possess superior information and are able to form (boundedly) rational expectations, while others are noise traders (see, for example, Shiller 1984, DeLong et al., 1990a, 1990b). Some models consider the choice between costly optimisation and cheap imitation strategies (see e.g. Sethi and Franke, 1995). In several other models artificial agents follow investment strategies based on standard theoretical principles, such as standard mean-variance optimisation (see Jacobs et al., 2004, Sharpe, 2007). Finally, it should be noted that contributions from econophysicists in this area of research are very significant – see Samanidou et al. (2007) for a review. A lot of agent-based financial models actually are a product of economists' close collaboration with physicists, drawing on the experience of the latter in studying systemic behaviour resulting from complex interaction of atomistic particles.

3.2.2. ASM models based on intelligent adaptation

Rational-expectations representative-agent models examine optimal investment strategies and asset pricing in the equilibrium. ASM models based on hard-wired investment strategies mostly deal with emergent properties of stock markets. All of this is important in ASM models based on intelligent adaptation. In addition to this, these ASM models also help to explain how investors may come up with good strategies, examine their stability and whether such artificial markets can generate equilibria derived from standard models with restrictive assumptions. Dynamically evolving and improving strategies is a remarkable feature of these agent-based financial market models. It greatly reduces reliance of modelled market behaviour on arbitrarily chosen investment strategies and increases model realism. Participants of real world financial markets act in a highly uncertain environment and most of them try to adapt to changing environmental conditions and learn to improve their strategies. Learning and adaptation mechanisms in ASM models may still be very far from anything that realistically describes genuine human learning but, in any case, attempts to model this crucial feature of investment behaviour constitute a qualitatively different approach to financial modelling. In most of ASMs, artificial intelligence techniques, most notably evolutionary algorithms and neural networks, are favoured over simplistic adaptive rules. Since intelligent adaptation implies choosing among many different strategies, as well as creating new ones, there is usually a large and evolving ecology of investment strategies in these models, and they are therefore sometimes referred to as the "many-type" models.

A detailed review of the related ASM literature is provided by LeBaron (2006), and here we only briefly mention some of the more popular models. A model proposed by Lettau (1997) is one of the early attempts to examine whether a population of heterogeneous agents are able to learn optimal investment strategies in a very simple market setting, for which the analytical solution is known. In his model, agents are endowed with myopic preferences and have to decide what fraction of their wealth to invest in the risky asset with exogenously determined random return. As is very common in ASM models, evolutionary systemic adaptation is ensured by the application of the genetic algorithm. Even though individual agents actually have no intelligence, fitter individuals (i.e. those whose strategies give higher levels of utility) have better chances of survival, and this evolutionary selection leads to near-optimal strategies over generations in this simple model. Routledge (1999, 2001) examines adaptive learning in financial markets in a more complex setting, namely, a version of Grossman and Stiglitz's (1980) model of heterogeneous information about future dividends and signal extraction. He presents analytic framework for adaptive learning via imitation of better-informed agents and shows that the rational expectations equilibrium is broadly supported by adaptation modelled with genetic algorithm.

Another interesting market setup is proposed by Beltratti and Margarita (1992). In this model, agents apply artificial neural network techniques to forecast stock prices from historical data, and trading may take place between randomly matched individuals with differing expectations. A noteworthy feature is that agents may choose to apply either a sophisticated neural network (with more hidden nodes, or explanatory variables) at a higher cost or a cheaper naïve network, and this corresponds to the real world fact that sophisticated investment is a costly endeavour. Interestingly, in some settings the naïve investors gain ground, once markets settle down and additional benefits from sophisticated forecasting do not cover its cost.

One of the most famous ASMs is the Santa Fe artificial stock market model developed by Arthur et al. (1997), also described in LeBaron et al. (1999) and LeBaron (2006). This model is aimed at exploring evolution and coexistence of a pool of strategies that compete with each other in the genetic algorithm environment and drive the market toward some informational efficiency. In this model there are two securities: a risky dividend-paying stock and a riskless bond that offers the constant interest rate. Heterogeneous agents have myopic[4] constant absolute risk aversion preferences. They try to forecast next period's stock price by applying simple "condition-forecast" rules and plug the forecast in their (induced) asset demand functions. The equilibrium price is determined by the auctioneer by balancing aggregate demand for shares with fixed supply. Forecast adaptation in this model is based on modified Holland's (1975) "condition-action" genetic classifier system. Each agent is given a set of rules mapping

[4] Agents care just about one period and do not dynamically optimise.

states of the world (such as the relative size of price/dividend ratio or a stock price relative to its moving average) to forecasts (which are linear combination of the current stock price and dividend). The rules endogenously evolve as a result of the cross-over, mutation and selection. The authors of the Santa Fe ASM examine convergence of the market price to the homogenous rational expectations equilibrium, and show that this is the case for certain parameter settings corresponding to the "slow learning" situation. The model is able to generate some statistical features of price dynamics qualitatively similar to stylized facts about the real world financial markets, though no attempt is made to quantitatively line them up with actual financial data or examine the realism of assumptions about dividend processes (LeBaron 2006). The model served as a platform for a number of further extensions (see e.g. Joshi et al., 2000, Tay and Linn, 2001, Chen and Yeh, 2001).

In an interesting model, LeBaron (2000) examines how investors' heterogeneous time horizons (i.e. different information sets, upon which agents choose decision rules) affect evolution of the market, its convergence to the known homogenous rational expectations equilibrium and domination of different types of investors. The model is in some respects similar to the Santa Fe model but has some notable original features. The learning mechanism in this model is an interesting combination of the neural network technique with the evolutionary search mechanism. In contrast to most other models, agents learn portfolio allocation decisions rather than explicitly form price expectations. Also, agents have access to a public – rather than private – pool of investment strategies, which are based on simple feedforward neural networks. Agents evaluate these strategies by feeding data series of heterogeneous length into these networks, and this forms the basis of their heterogeneity. Furthermore, the neural networks are evolved by applying mutation, crossover, weight reassignment and rule removal operations, and some of the worst-performing individuals are also replaced by new agents. One of the model's main findings is that short-memory (and short investment horizon) investors are not driven out of the market, and acting as some market volatility generators they hinder attaining the rational expectations equilibrium.

4. Reinforcement learning in the context of ASM modelling

Expectations regarding prospects of a particular stock or a stock index play a crucial role in active portfolio management. If one is willing to accept the obvious empirical fact of a large variety of market expectations, or if one is aimed at explaining how this diversity comes about, then it is necessary to abandon the rational expectations assumption[5], which allowed to circumvent these issues in standard models. According to Arthur (1995), rational expectations equilibrium is a special and in many cases not robust state of reality, whereby individual expectations induce actions that aggregatively create a world that validates them as predictions.

But how do investors behave in realistic, out-of-equilibrium situations? Needless to say, realistic modelling of expectations formation poses great challenge and it is essentially a grey area of the financial theory. A few observations about investor

[5] The rational expectations equilibrium can only be warranted if the critical mass of agents basically apply *the same*, true model of the reality (possibly with some non-systemic perception errors).

behaviour in a highly uncertain environment can be made, however. In such an environment it seems perfectly sensible for investors to act adaptively and follow inductive reasoning, or in other words, form simple forecasting models, test them and update them depending on their performance. Hence, investors should constantly learn from interaction with environment. Their learning is not a supervised process because due to the model uncertainty generally there is no way of knowing the true intrinsic value of a stock even in retrospective. For instance, if an investor observes a stock price realisation, which is different from what he had expected, he generally cannot know whether this deviation is attributable to his misperception of fundamentals, unpredictable shocks to fundamentals, complex interaction of market participants or other factors. However, even without knowing retrospectively what the "correct" expectations and actions should have been, investors can judge about adequacy of their beliefs and actions by reinforcement signals that they receive from interaction with the environment. Possible reinforcement signals include their performance relative to the market, long-term portfolio returns, utility from consumed earnings, etc. Another important aspect is that in order to better adapt to the uncertain environment investors have to both exploit their accumulated experience and explore seemingly suboptimal actions.

If the above-described adaptive investment behaviour is deemed an adequate description of how boundedly rational investors actually behave, reinforcement learning methods developed in the artificial intelligence literature seem to be conceptually suitable for modelling investor behaviour (though there are some problems with technical implementation).

Inspired by the psychology literature, reinforcement learning is the sub-area of the machine learning, and in its core lies agents' interaction with environment in pursuit of highest long-term rewards. It could be of a particular interest to economists because standard theoretical economic agents' behaviour is often guided by very similar principles. In standard economics and finance, agents choose action plans that ensure maximisation of life-time utility (long-term rewards), and that is exactly what reinforcement learning agents seek – the main difference being that the latter do not know the underlying model of the economy. The well-established link between the basic reinforcement learning algorithm and dynamic programming, as well as the proven ability of some reinforcement learning algorithms to achieve (under certain conditions) convergence to optimal policies are especially attractive features of this methodology, from the economists' viewpoint. The reinforcement learning agents are also well-positioned to solve the temporal credit assignment problem, i.e. determine strategic actions that enable them to reach their ultimate goals even though those actions may not give desirable results in the short-term. For economists, it is a great advantage over simpler forms of adaptive learning.

Reinforcement learning addresses the question of how an autonomous agent that senses and acts in its environment can learn to choose optimal actions to achieve its goals (Mitchell 1997, p. 367). By taking actions in an environment and obtaining immediate associated rewards, a reinforcement learning agent tries to find optimal policies, which maximise long-term rewards, and the process of improvement of agent policies is the central target for reinforcement learning methods[6].

Practical implementation of reinforcement learning techniques in the domain of interesting economic and financial problems can be problematic, as policy convergence requirements include stationary environment, fully observable states and the single-agent setting. In other words, the reinforcement learning agent is capable of learning to effectively adapt in the well-defined stationary environment but, naturally, simple adaptive learning cannot guarantee optimal behaviour once the multi-agent interaction brings in strong non-stationarity and strategic interaction among agents.

Existing financial research provides little guidance on applying reinforcement learning ideas in the ASM context. To our knowledge, no well-known multi-agent stock market models based on reinforcement learning have been developed so far. The reinforcement learning literature, however, provides some evidence that using the single-agent reinforcement learning (more specifically, Q-learning) algorithm in the multi-agent setting quite often leads to either exactly or approximately optimal policies (Tesauro, 2002). For instance, Tesauro and Kephart (2002) show that in a stylised two-seller market price-setting policies derived by using the standard Q-learning algorithm outperform some fixed and myopic policies and, in some settings, convergence to optimal policies is achieved. It should also be noted that in order to improve performance in multi-agent settings different extensions to the standard reinforcement learning algorithms have been proposed. They are mainly applied in two-player games, and they take into account the opponent's estimated strategies (e.g. Littman's (1994) Minimax-Q algorithm, Tesauro's (2004) Hyper-Q algorithm or the Nash-Q algorithm developed by Hu and Wellman, 2003). Alternatively, agents may adapt learning rates according to the current performance, as in WoLF (Win or Learn Fast) algorithm developed by Bowling and Veloso (2001).

As the current discussion paper is an integral part of our ongoing research aimed at combining ASM modelling with the reinforcement learning techniques, we provide a cursory discussion how we deal with the abovementioned implementation problems. In our research, strategic interaction among agents is limited as agents interact in a competitive manner via the centralised exchange, where they participate in double auctions and take decisions simultaneously. The problem is arguably alleviated by the fact that the number of agents is quite large, and what matters for any specific agent is the relatively stable distribution of all other agents' actions (buy or sell orders) rather than individual actions per se. Hence, the average market price and other trading statistics can be seen as some summarising functions of multi-agent interaction and this is taken into account when individual decisions are made. Additional stability of learning processes can also be ensured by combining the Q-learning algorithm with some evolutionary adaptation principles.

It should also be noted that reinforcement learning methods have been quite successfully applied in various portfolio management problems. One of the early applications is Neuneier's (1996) model, which employs the Q-learning in combination with the neural network as a value function approximator for optimal currency

[6] A good introduction to the reinforcement learning techniques may be found in Sutton and Barto (1998), Bertsekas and Tsitsiklis (1996) and Mitchell's (1997) books, and some broad overview of reinforcement learning models is given in Kaelbling, Littman and Moore (1996) survey.

allocation in a simple two-currency, risk-neutral setting. Moody and Saffell (2001) apply direct reinforcement learning to optimise risk-adjusted investment returns on the intra-day currency and stock-index trading. Van Roy (1999) uses the temporal difference learning algorithm for valuing financial options and optimising investment portfolio. Reinforcement learning methods of portfolio management are gradually gaining popularity among practitioners but theoretical literature remains relatively scarce and further research is needed to unleash the potential of this approach.

5. Concluding remarks

Global economic and financial problems caused by the disorderly unwinding of imbalances that had been accumulating over decades cannot be easily reconciled with the efficient market paradigm based on rational expectations and perfectly rational representative agent assumptions. Some even see the collapse of global consumption as a result of the disregard of intertemporal budget constraints, which is of course at odds with basic principles of consumer (investor) optimising behaviour. In any case, the mainstream financial and economic theory cannot explain current economic developments, give adequate forecasts or policy prescriptions. In this situation there is actually a growing need to look for a replacement for the work-horse financial and macroeconomic models based on heroic assumptions. This paper advocates the view that agent-based financial modelling, which models financial markets as complex dynamical systems consisting of interacting heterogeneous agents, can potentially become a viable alternative. However, significant further interdisciplinary progress is needed before agent-based models can take the centre-stage of financial market modelling.

In the paper we discuss, at the conceptual level, the need for the proper account of market complexity, agent heterogeneity, bounded rationality and adaptive (though not simplistic) expectations. These features are usually absent in standard financial models but are at the heart of agent-based modelling. Agent-based models use procedural (computer code) rather than functional (mathematical equations) model description form, which technically enables researchers to easily inquire into many interesting features of systemic behaviour. However, theoretical or empirical foundation of individual agents' behaviour and external validation of model results are two main problem areas, which have not been systemically addressed to date and constitute a serious obstacle to the further progress of agent-based financial modelling.

The paper focuses in particular on the artificial stock market modelling. In the brief literature review two broad categories of models are presented: (*i*) models based on stochastic, heuristic and standard theory-implied behavioural rules and (*ii*) models with learning agents or evolutionary systemic adaptation. The first group of models generally emphasise the role of agent heterogeneity in determining complex market behaviour and emergent systemic properties, whereas the second group of models concentrate on the generation of good investment strategies and the macro-level implications of intelligently adapted investment strategies.

Intelligent adaptation in the highly uncertain environment can arguably be key to understanding actual financial market behaviour. It is instructive to resort to the artificial intelligence literature for specific algorithms of adaptation and learning. In particular, we find reinforcement learning algorithms economically very appealing, though there are certain problems with their practical implementation. Without knowing the true model of reality, reinforcement learning agents learn from interaction with environment and adjust their strategies so that they attain maximum long-term reinforcement (utility) from the environment. Similarly, investors seek to reach their long-term objectives in highly uncertain environment. We continue to explore these ideas further in the ongoing artificial stock market research.

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