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The Factor Analytical Approach in Trending Near Unit Root Panels [1]

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ABSTRACT

In this study, we re-visit the factor analytical (FA) approach for (near unit root) dynamic panel data models, whose asymptotic distribution has been shown to be normal and well centered at zero without the need for valid instruments or correction for bias. It is therefore very appealing. The question is: Does the appeal of FA, which so far has only been documented for fixed effects panels, extends to panels with incidental trends? This is an important question, because many persistent variables are trending. The answer turns out to be negative. In particular, while consistent, the asymptotic normality of FA breaks down when there is an exact unit root present, which limits its applicability.

Keywords: Dynamic panel data models, Unit root, Factor analytical method

JEL codes: C12, C13, C33

1 Introduction

In this paper, we consider the following panel data model, which is the kernel of most models in the dynamic panel data literature (see Baltagi, 2008):

$$y_{i,t} = \lambda_i' D_t + z_{i,t}, \quad (1.1)$$

$$z_{i,t} = \rho z_{i,t-1} + \varepsilon_{i,t}, \quad (1.2)$$

where $i = 1, ..., N$ and $t = 1, ..., T$ index the cross-sectional units and time periods, respectively, $z_{1,0} = ... = z_{N,0} = 0$, $\varepsilon_{i,t}$ is an error term, D_t is vector of deterministic trend terms, and λ_i is a conformable vector of coefficients. We assume that $\varepsilon_{i,t}$ is independent and identically distributed across both i and t with $E(\varepsilon_{i,t}) = 0$ and $E(\varepsilon_{i,t}^2) = \sigma^2 > 0$, and that $S_\lambda = N^{-1} \sum_{i=1}^N \lambda_i \lambda_i' \to \Sigma_\lambda$ as $N \to \infty$, where Σ_λ is positive definite. These conditions are restrictive but they can be relaxed along the lines of Bai (2013), and Norkutė and Westerlund (2021) to accommodate, for example, non-zero initial values, exogenous regressors, heteroskedasticity and weak serial correlation in $\varepsilon_{i,t}$.[1] The autoregressive parameter, ρ, is assumed to be "local-to-unity" in the following sense:

$$\rho = \exp(cN^{-\eta}T^{-\gamma}) = 1 + c\alpha T^{-1} + O(\alpha^2 T^{-2}) \quad (1.3)$$

where $c \in \mathbb{R}$ is a local-to-unity parameter, $\alpha = \alpha(N, T) = N^{-\eta}T^{1-\gamma}$ with $\lim_{N,T\to\infty} \alpha = \alpha_0 \in [0, \infty)$, and $\eta \geq 0$ and $\gamma \geq 0$ determine the rate at which $\rho \to 1$. This formulation is very flexible and includes most previously considered local specifications as special cases, such as the usual time series setup with $\eta = 0$ and $\gamma = 1$, and the common fixed effects panel data specification with $\eta = 1/2$ and $\gamma = 1$ (see Westerlund and Larsson, 2015, for a discussion).

[1]The restrictive conditions are there to ensure a manageable asymptotic analysis, and are not necessary in practice, provided that the FA estimator is suitably modified. The zero initial value condition is particularly simple to relax, as the FA estimator is actually asymptotically invariant to the values taken by $z_{1,0}, ..., , z_{N,0}$, provided that they are $O_p(1)$. Cross-section heteroskedasticity can also be accommodated without change, provided that $N^{-1} \sum_{i=1}^N E(\varepsilon_{i,t}^2)$ has a limit such as σ^2. For discussions of how to accommodate exogenous regressors, time heteroskedasticity and weak serial correlation in $\varepsilon_{i,t}$, we make reference to Norkutė and Westerlund (2021).

The estimation of ρ has attracted a considerable attention in the literature. One of the main reasons for this is the problems caused by the presence of the incidental parameters in $\lambda_1, ..., \lambda_N$. The standard approach is to de-trend the data prior to estimation. However, this makes the lagged dependent variable correlated with the error term, which in turn complicates estimation and inference. This is true not only in the classical dynamic panel data setup with T fixed, but also in the type of large N and T panels considered here. The main concern is that the asymptotic distributions of many known estimators, such as the least squares (LS) estimator, are not correctly centered at zero, which invalidates inference. Generalized method of moments (GMM) estimators are an alternative, but they tend to suffer from weak instrumentation problems when $\rho \to 1$ (see Roodman, 2009). Another alternative is to use bias-corrected estimators, such as the bias-corrected LS estimator of Hahn and Kuersteiner (2002). However, the appropriate correction depends not only on the elements of D_t, but potentially also on unknown nuisance parameters (see Moon and Phillips, 2000). Moreover, performance can be very sensitive to the way the correction is carried out, so much so that some researchers have cautioned against its use (see Moon and Perron, 2004).

The FA estimator of Bai (2013) does not require estimation of $\lambda_1, ..., \lambda_N$ but only estimation of S_λ, which is a finite-dimensional object. As a result, the estimator attains a normal asymptotic distribution that is correctly centered at zero despite being completely instrument- and correction-free. This makes it very attractive from both theoretical and empirical points of view. Bai (2013) considers the case when $|\rho| < 1$, but Norkutė and Westerlund (2021) have shown that the attractiveness of FA applies also under (1.3). One implication of this is that FA can be used for unit root testing, and Norkutė and Westerlund (2021) have shown that it is possible to construct FA-based unit root tests with maximal achievable power.

As with the bulk of the existing literature, Bai (2013), and Norkutė and Westerlund (2021) focus on the case when $D_t = 1$. Of course, for many economic time series, a constant and linear trend, rather than just a constant, is likely to be the appropriate deterministic specification. This is certainly true for variables such as GDP, industrial production, money supply and consumer or commodity prices, where trending behavior is evident. The present paper is motivated by this observation. The purpose is to study the properties of FA when $D_t = (1, t)'$. The main finding is that while when $c \neq 0$ the asymptotic distribution of FA is normal and well centered at zero, when $c = 0$ the Hessian of the FA objective function is zero asymptotically, and therefore asymptotic normality breaks down. The estimator is still consistent, but the zero Hessian means that it is not suitable for unit root testing, and that it is likely to be subject to numerical optimization problems. The introduction of the trend therefore removes much of the appeal of FA.

2 The FA estimator

Let us write (1.1) and (1.2) as

$$y_{i,t} = d_{i,t} + \rho y_{i,t-1} + \varepsilon_{i,t}, \tag{2.1}$$

where $d_{i,t} = \lambda_i'(D_t - \rho D_{t-1})$ for $t \geq 2$ and $d_{i,t} = \lambda_i' D_t$ for $t = 1$. This equation can be written on stacked vector notation;

$$y_i = d_i + \rho J y_i + \varepsilon_i, \tag{2.2}$$

where $y_i = [y_{i,1}, ..., y_{i,T}]'$, $d_i = [d_{i,1}, ..., d_{i,T}]' = (I_T - \rho J)D\lambda_i$ and $\varepsilon_i = [\varepsilon_{i,1}, ..., \varepsilon_{i,T}]'$ are $T \times 1$, $D = [D_1, ..., D_T]'$ is $T \times 2$ and J is the $T \times T$ lag matrix with ones just below the main diagonal and zeros elsewhere. The above equation can be solved for y_i, giving

$$y_i = \Gamma(\rho) d_i + \Gamma(\rho)\varepsilon_i, \tag{2.3}$$

where $\Gamma(\rho) = (I_T - \rho J)^{-1} = I_T + \rho L(\rho)$ and

$$L(\rho) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \rho & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \rho^{T-2} & \dots & \rho & 1 & 0 \end{bmatrix}. \tag{2.4}$$

While in the supplemental material, we treat σ^2 as unknown, for simplicity here we follow Moon and Phillips (1999), and treat it as known. The vector of parameters of interest is therefore given by $\theta = [(\text{vech}\, S_\lambda)', \rho]' = (\theta_1', \theta_2')'$, where $\theta_1 = \text{vech}\, S_\lambda$, $\theta_2 = \rho$, and vech is the half-vec operator. The purpose of FA is to make inference regarding this vector. This is done by considering the following "discrepancy" function (see Bai, 2013, for a discussion):

$$Q(\theta) = \log(|\Sigma(\theta)|) + \text{tr}\,[S_y \Sigma(\theta)^{-1}], \tag{2.5}$$

where $|A|$ and $\text{tr}\, A$ are the determinant and trace, respectively, of A, $S_y = N^{-1}\sum_{i=1}^N y_i y_i'$, $\Sigma(\theta) = \sigma^2\Gamma(\rho)\Lambda(S_\lambda, \sigma^2)\Gamma(\rho)'$ and $\Lambda(S_\lambda, \theta_2) = I_T + \Gamma(\rho)^{-1} D S_\lambda D' \Gamma(\rho)^{-1\prime}$. The objective function is given by $\ell(\theta) = -NQ(\theta)/2$. Let $Q^*(\rho) = \max_{S_\lambda} Q(\theta)$. In the supplement, we show that the maximizer with respect to S_λ is given by

$$\hat{S}_\lambda(\rho) = [\Gamma(\rho)^{-1}D]^{-}(G(\rho) - \sigma^2 I_T)[\Gamma(\rho)^{-1}D]^{-\prime}, \tag{2.6}$$

where $A^{-} = (A'A)^{-1}A'$ for any full row rank matrix A and $G(\rho) = \Gamma(\rho)^{-1}S_y\Gamma(\rho)^{-1\prime}$. Let $\hat{\Lambda}(\rho) = I_T + \sigma^{-2}\Gamma(\rho)^{-1}D\hat{S}_\lambda(\rho)D'\Gamma(\rho)^{-1\prime}$. Concentration and simplification lead to

$$Q^*(\rho) = T\log(\sigma^2) + \log(|\hat{\Lambda}(\rho)|) + \sigma^{-2}\text{tr}\,[G(\rho)\hat{\Lambda}(\rho)^{-1}], \tag{2.7}$$

with

$$\ell^*(\rho) = -\frac{N}{2}Q^*(\rho) \tag{2.8}$$

being the concentrated version of $\ell(\theta)$. The objective is to maximize $\ell^*(\rho)$ with respect to ρ. Let us therefore denote the true value of c by c_0. The true value of ρ is given by $\rho_0 = \exp(c_0 N^{-\eta}T^{-\gamma})$. The FA estimator $\hat{\rho}$ of ρ_0 is defined as

$$\hat{\rho} = \arg\max_{\rho\in\mathbb{R}} \ell^*(\rho). \tag{2.9}$$

3 Results

We start by considering the issue of consistency. In the supplement, we show that

$$Q^*(\rho) = q(c) + o(1), \tag{3.1}$$

where $q(c)$ is a continuous function whose exact definition is extremely lengthy and is therefore relegated to the supplement. The $o(1)$ remainder is uniform in c. The fact that $q(c)$ is written as a function of c (and not of ρ) involves no loss of generality, because asymptotically $q(c)$ is minimized at $\hat{c} = N^\eta T^\gamma(\hat{\rho} - 1)$ (see Moon and Phillips, 2004). However, it simplifies the analysis, because unlike ρ_0, c_0 is a fixed parameter that does not tend to zero. Note in particular that since c_0 is an interior point and since $q(c)$ is differentiable, if c_0 minimizes $q(c)$, it must be that $dq(c_0)/dc = 0$ (see Moon and Phillips, 1999, page 723). We therefore proceed to differentiate $q(c)$.

Lemma 1.

$$\begin{aligned}\frac{d}{dc}q(c) &= -\frac{1}{h_1(c)^2}[(c_0 - c)\alpha^2 q_2 + (c_0 - c)^2\alpha^3 q_3 + (c_0 - c)^3\alpha^4 q_4 \\ &+ (c_0 - c)^4\alpha^5 q_5 + (c_0 - c)^5\alpha^6 q_6],\end{aligned}$$

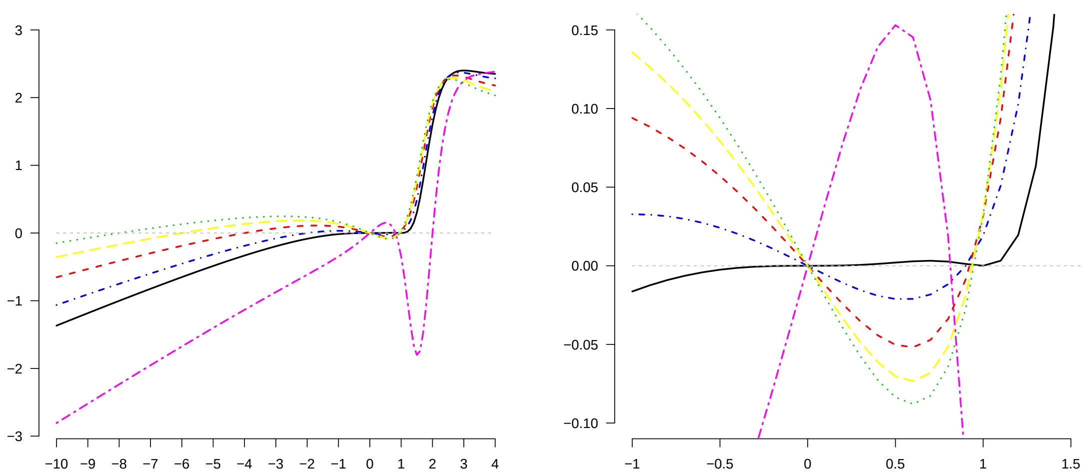
where $h_1(c)$, q_2, q_3, q_4, q_5 and q_6 are given in the supplement.

The roots of $dq(\hat{c})/dc = 0$ can be obtained by using numerical methods, such as Euler's method. In this paper, however, we follow Moon and Phillips (2004), and use Mathematica 10.1's command Solve, which has the advantage that the roots can be obtained analytically. What we find is that only three out of the five roots are real. We therefore focus on these, as $\rho_0 \in \mathbb{R}$.

Figure 1: $dq(c)/dc$ for different values of c_0 and c when $\alpha = 1$.

(a) Zoom out

(b) Zoom in



Note: The horizontal axis represents values of c. The black (solid), pink (long dash-dotted), blue (dash-dotted), red (dashed), yellow (long dashed) and green (dotted) lines are for c_0 equal to 0, 2, -2, -4, -6 and -8, respectively.

Figure 1 plots $dq(c)/dc$ for different values of c_0 and c when $\alpha = 1$; later on we comment on how the results are affected when varying α. The first two roots are given by $c = 0$ and $c = c_0$. The third root involves some serious complexity of expression and is therefore not reported here. Figure 1 suggests that it lies somewhere in the interval $[0.5, 1]$. However, the global minimizer is always given by $c = c_0$, which implies that $\hat{c}$ is consistent, and therefore so is $\hat{\rho}$ (see Section 4 in Moon and Phillips, 2004, for a similar argument). Theorem 1 formalizes this.

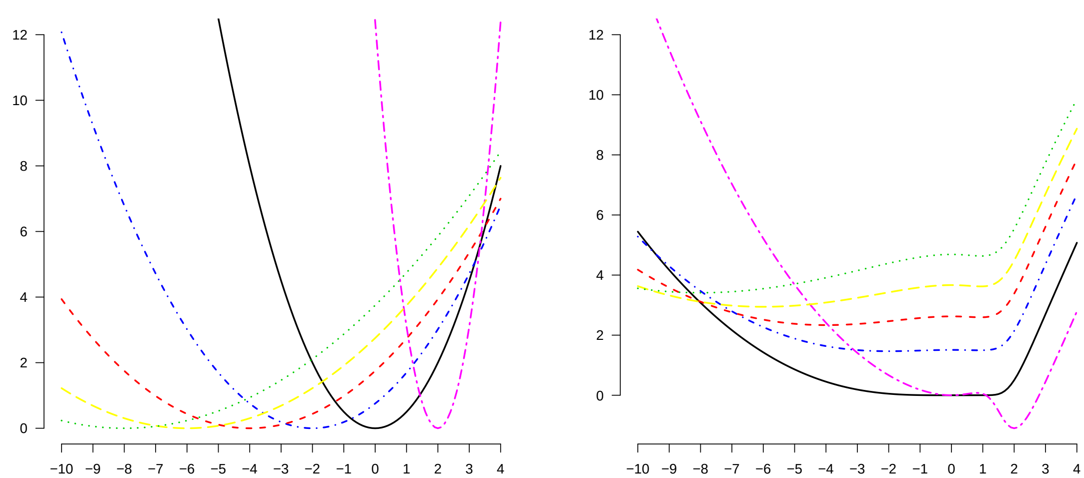
Theorem 1. *As N, $T \to \infty$,*

$$\hat{\rho} \to_p \rho_0.$$

The fact that $\hat{c}$ is consistent when there is a linear trend present stands in sharp contrast to previous results. Moon and Phillips (1999) show that the maximum likelihood estimator of c_0 is inconsistent, and that this is due to the fact that the score of the log-likelihood function has a nonzero mean in the limit. The normal equation of the pooled LS estimator of c_0 based on the detrended data is also biased (Moon and Phillips, 2000). In fact, the GMM estimator of Moon and Phillips (2004) is the only other estimator known to us that allows consistent estimation of c_0 under a linear trend.

Figure 2: $q(c)$ for different values of c_0 and c when $\alpha = 1$.

(a) $D_t = 1$ (b) $D_t = (1, t)'$



Note: See the explanation of Figure 1.

Of course, consistency does not mean that the FA estimator is free of complications. To see this, Figure 2 plots $q(c)$ for both $D_t = 1$ and $D_t = (1, t)'$. Unlike when $D_t = 1$, we see that when $D_t = (1, t)'$ the global minimum at $c = c_0$ gets closer to the local maximum at $c = 0$ as c_0 approaches zero. The reason is that under a linear trend $q(c)$ is not globally but only locally convex around the global minimum c_0, and the convexity of $q(c)$ around c_0 is decreasing in c_0. This means that the estimation is likely to be more difficult the closer c_0 is to zero. Figure 2 illustrates this point for $\alpha = 1$; however, the same is true also for $\alpha \neq 1$, as we illustrate in the supplement. The main conclusion from varying α is just as expected given the effect of c_0 reported in Figures 1 and 2, and how c_0 and α enter (1.3) multiplicatively; that is, the global minimum is easier (harder) to discern the larger (smaller) is α, as the convexity of $q(c)$ around c_0 is increasing in α.

The above mentioned difficulty in finding the global minimum is reflected in the asymptotic distribution of $\hat{\rho}$. Theorem 2 and Corollary 1 report this asymptotic distribution for $\alpha_0 > 0$ and $\alpha_0 = 0$, respectively. Both results assume that $c_0 \neq 0$; however, we also discuss the case when $c_0 = 0$.

Theorem 2. *Suppose that $\alpha_0 > 0$ and $c_0 \neq 0$. Then, as $N, T \to \infty$,*

$$\sqrt{N}T(\hat{\rho} - \rho_0) \to_d N\left(0, \lim_{N,T\to\infty} \frac{1}{s^2(\rho_0)}\right),$$

where

$$s^2(\rho_0) = -\frac{1}{NT^2}\frac{d^2\ell^*(\rho_0)}{(d\rho)^2} = \frac{\alpha_0^2 c_0^2}{45} + O(\alpha_0^3 c_0^3).$$

According to Theorem 2, there is no asymptotic bias despite the linear trend and local-to-unity specification of ρ. This is important, because, as alluded to in Section 1, most existing estimators of ρ_0 are biased in ways that depend on the fitted deterministic specification. Valid inference in these cases therefore requires bias correction, which can sometimes be detrimental for performance (see Moon and Perron, 2004).

A major problem revealed by Theorem 2 is that

$$\lim_{N,T\to\infty} s^2(1) = 0. \tag{3.2}$$

Hence, since the Hessian is zero, its inverse, which is identically the asymptotic variance of $\hat{\rho}$, is undefined when $c_0 = 0$ and/or $\alpha_0 = 0$, and therefore ρ_0 is first-order unidentified (see Moon and Phillips, 2004). This is partly expected given Figure 1, which shows that $dq(c)/dc$ is flat in a neighborhood around $c_0 = 0$. Hence, while optimal in the constant only case (see Section 1), with a trend included FA is not really suitable for unit root testing, as $c_0 = 0$ under the null hypothesis. Higher-order identification might be possible, but even if it is the results of Moon and Phillips (2004) suggest that the rate of convergence is not likely to be larger than $N^{1/6}T$ and the asymptotic distribution may be non-standard, and our preliminary Monte Carlo results support this. We therefore do not pursue this avenue any further, but rather we caution against the use of FA as a basis for constructing unit root tests when there is a linear trend present.

Because the rate of shrinking of the local specification in (1.3) is decreasing in α, we might expect $\alpha_0 = 0$ to lead to relatively low rate of convergence when compared to $\alpha_0 > 0$. Corollary 1 confirms this.

Corollary 1. *Suppose that $\alpha_0 = 0$ and $c_0 \neq 0$. Then, as N, $T \to \infty$ with $\max\{N^{-1/2}, T^{-1/2}\}/\alpha \to 0$,*

$$\alpha\sqrt{N}T(\hat{\rho} - \rho_0) \to_d N\left(0, \frac{45}{c_0^2}\right).$$

The case considered in Corollary 1 with $\alpha_0 = 0$ and $c_0 \neq 0$ is important, as it is consistent with most local alternatives considered in the unit root testing literature. In the trend case considered here, the power envelope is defined within $N^{-1/4}T^{-1}$-neighborhoods of the unit root null hypothesis (Moon et al., 2007), which in our notation is tantamount to setting $\gamma = 1$ and $\eta = 1/4$, such that $\alpha = N^{-1/4}$ and therefore the rate of convergence of $\hat{\rho}$ is given by $\alpha\sqrt{N}T = N^{1/4}T$.[2] Hence, provided that $c_0 \neq 0$, the FA estimator operates within the optimal shrinking neighborhood.

An important implication of the results reported so far for empirical work is that while $\hat{\rho}$ is consistent, implementation may be difficult as conventional gradient-based optimization methods are likely to fail if c_0 is close to zero. In order to illustrate this point, in Table 1 we report the bias and root mean squared error (RMSE) of $\hat{\rho}$ when $\gamma = 1$, $\eta = 1/4$, $\varepsilon_{i,t} \sim N(0, 1)$ and the elements of λ_i are drawn independently from $U(0, 1)$. The optimization was carried out using the BFGS algorithm with the true parameters as starting values, which performed very similarly to Newton–Raphson. The number of replications is 1,000. As expected given Theorem 1, we see that both the bias and RMSE are decreasing in N and T. The fact that the decrease is faster in T than in N is consistent with the rate of convergence given in Theorem 2.

As in Hsiao et al. (2002), the results reported in Table 1 are based on the replications in which FA could be computed. In order to get a feeling for the numerical performance of FA, in Table 2 we report the number of replications with either complete break-down

[2]The expansion of the score used to derive the asymptotic distribution in Theorem 2 is accurate up to an $O_p(\max\{N^{-1/2}, T^{-1/2}\})$ remainder. The Corollary 1 requirement that $\max\{N^{-1/2}, T^{-1/2}\}/\alpha \to 0$ is there to ensure that this remainder remains negligible even when the score is scaled by α. Note in particular how $N^{1/4}T$-consistency requires $N^{1/4}T^{-1/2} \to 0$, which is in agreement with the results reported by Moon et al. (2007).

or non-invertible Hessian. As expected, we see that the numbers increase as c_0 becomes closer to zero. The fact that the Hessian can be non-positive definite means that the FA-based t-statistic can sometimes take on complex numbers, which in turn makes it difficult to evaluate its size. The results do get better as c_0 moves away from zero, but then these are not very interesting since in practice $c_0 = 0$ is main hypothesis of interest.[3] The results reported in Tables 1 and 2 are for the case when the starting values set equal to the true parameters, and, as expected, the performance is even worse if FA is initialized at the LS estimates. Hence, not only is FA likely to be difficult to implement in practice, but it can also be quite uninformative.

Table 1: Bias and RMSE.

		$c_0 = 0$		$c_0 = -2$		$c_0 = -4$		$c_0 = -6$	
N	T	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
50	50	−0.0023	0.0179	−0.0018	0.0194	0.0008	0.0196	−0.0038	0.2683
100	50	−0.0012	0.0144	−0.0007	0.0150	0.0015	0.0153	0.0007	0.0133
200	50	−0.0010	0.0121	−0.0006	0.0119	0.0013	0.0119	0.0005	0.0098
50	100	−0.0010	0.0092	−0.0001	0.0100	0.0009	0.0100	0.0004	0.0089
100	100	−0.0007	0.0074	−0.0001	0.0079	0.0010	0.0081	0.0004	0.0067
200	100	−0.0005	0.0064	−0.0002	0.0062	0.0009	0.0064	0.0003	0.0052
50	200	−0.0007	0.0047	−0.0002	0.0049	0.0005	0.0050	0.0001	0.0044
100	200	−0.0004	0.0039	0.0000	0.0040	0.0004	0.0039	0.0001	0.0033
200	200	−0.0002	0.0033	0.0001	0.0033	0.0004	0.0033	0.0001	0.0026

Notes: c_0 is such that $\rho_0 = 1 + c_0 N^{-1/4} T^{-1}$.

[3]We have computed rejection rates based on the Monte Carlo iterations that "worked", and they are close to 5%, provided that c_0 is sufficiently far away from zero. Instead of the inverse Hessian, one may use the following analytical plug-in variance estimator: $s^2(\hat{\rho}) = T^{-2}\text{tr}\,[(L(\hat{\rho})' + L(\hat{\rho}))M_{\Gamma(\hat{\rho})^{-1}D} L(\hat{\rho}) M_{\Gamma(\hat{\rho})^{-1}D}]$, where $M_A = I - A(A'A)^{-1}A'$ for any matrix A. This alternative estimator is better behaved than the inverse Hessian in the sense that it is typically positive. However, because asymptotic normality breaks down when $c_0 = 0$, the better behaviour only matters when c_0 is far away from zero. Hence, regardless of the variance estimator used, the FA-based t-test is not very useful, because the main hypothesis of interest is again given by $c_0 = 0$.

Table 2: Numerical diagnostics.

		$c_0 = 0$		$c_0 = -2$		$c_0 = -4$		$c_0 = -6$	
N	T	Hess	Break	Hess	Break	Hess	Break	Hess	Break
50	50	3.9%	0.8%	3.1%	0.1%	2.4%	0.0%	0.4%	0.0%
100	50	7.3%	0.3%	6.6%	0.1%	3.6%	0.0%	1.3%	0.0%
200	50	9.8%	0.1%	11.6%	0.0%	9.5%	0.0%	1.8%	0.0%
50	100	3.4%	0.5%	2.5%	0.0%	1.3%	0.0%	0.2%	0.0%
100	100	6.9%	0.1%	5.9%	0.0%	4.0%	0.0%	0.4%	0.0%
200	100	10.8%	0.1%	11.8%	0.1%	7.2%	0.0%	1.6%	0.0%
50	200	4.1%	0.2%	3.5%	0.0%	2.6%	0.0%	0.4%	0.0%
100	200	7.3%	0.4%	5.2%	0.1%	3.5%	0.0%	0.4%	0.0%
200	200	10.5%	0.1%	9.2%	0.0%	7.8%	0.0%	0.9%	0.0%

Note: “Hess” and “Break” refer to the fraction of 1000 replications in which the Hessian is non-positive definite, and the number of break-downs, respectively. c_0 is such that $\rho_0 = 1 + c_0 N^{-1/4} T^{-1}$.

4 Conclusion

In this paper, the FA estimator of Bai (2013), originally proposed for stationary dynamic panels with fixed effects, is applied to near unit root panel data with a linear trend. What we find is that while the estimator is consistent and asymptotically normal, the rate of consistency depends on the closeness to the unit root, and asymptotic normality breaks down when $c_0 = 0$, which is a problem because a unit root is also the main hypothesis of interest. The break-down is likely to affect both the implementation and performance of FA not only when c_0 is at zero, but when it is “close” to zero, and our Monte Carlo results confirm this. For these reasons, we caution against the use of FA when there is a trend present.

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Supplement to "The Factor Analytical Approach in Trending Near Unit Root Panels": Omitted Results and Proofs

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Abstract

This supplement we (i) summarize the assumptions used in the main paper, (ii) study the effect of α on the roots of $q(c)$, (iii) provide the derivation of the concentrated objective function, and (iv) provide all the proofs that were omitted from the main paper.

A Assumptions

The following assumptions generalize those used in the main paper to the case when σ^2 (and its true value σ_0^2) is unknown.

Assumption 1. $\varepsilon_{i,t}$ is independent and identically distributed (iid) across both i and t with $E(\varepsilon_{i,t}) = 0$, $E(\varepsilon_{i,t}^2) = \sigma^2 \in \mathbb{S}$, $\mathbb{S} = [\underline{\sigma}^2, \overline{\sigma}^2] \subset \mathbb{R}_{++}$, $0 < \underline{\sigma}^2 < \overline{\sigma}^2 < \infty$ and $\sigma^{-4}E(\varepsilon_{i,t}^4) = \kappa < \infty$, where $\mathbb{R}$ is the set of real numbers.

Assumption 2. $c \in \mathbb{C}$, where $\mathbb{C} = [\underline{c}, \overline{c}] \subset \mathbb{R}$ and $-\infty < \underline{c} < \overline{c} < \infty$.

Assumption 3. $\alpha = \alpha(N, T) = N^{-\eta}T^{1-\gamma} \to \alpha_0 \in \mathbb{A} = [0, \infty)$ as $N,\ T \to \infty$.

Assumption 4. $S_\lambda = N^{-1}\sum_{i=1}^{N} \lambda_i\lambda_i' \to \Sigma_\lambda$ as $N \to \infty$, where Σ_λ is positive definite.

Assumption 5. θ_2^0 lies in the interior of $\Theta_2 = \mathbb{R} \times \mathbb{S}$.

Assumptions 1–4 are the same as in the main paper. Assumption 5 is necessary once σ^2 is included in the analysis.

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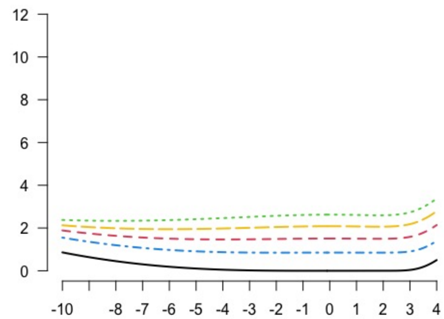
B The effect of α on the roots of $q(c)$

Figure B.1 plots $q(c)$ for different values of c_0 and α. Here the horizontal axis measures the values of c. As in the main text, the black (solid), blue (dash-dotted), red (dashed), yellow (long dashed) and green (dotted) lines represents c_0 equal to 0, -2, -4, -6 and -8, respectively. The parameter α serves a similar purpose as c_0 regulating the closeness to the unit root ($c_0 = 0$). In particular, we see that for any c_0, as α increases (decreases), $q(c)$ gains (looses) curvature and becomes more convex. For large values of α, $q(c)$ becomes close to a globally convex function. We also see that as α increases the local maximum at $c = 0$ gets closer to the local minimum at the third root. However, as long as c_0 sufficiently far away from zero, this is not problematic, because with an increase in α the global minimum at c_0 becomes easier to distinguish from the local stationary points due to convexity. Therefore, optimization of $q(c)$ is easier and the overall performance of FA in the trend case is better the larger is α.

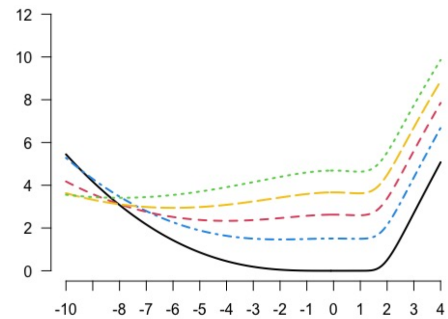
When c_0 is "close" to zero, the global minimum at c_0 and the local maximum at $c = 0$ are close. Moreover, the local minimum at the third root is close to c_0 and therefore $q(c)$ flattens out around c_0 as can be seen from (a)–(f) in Figure B.1.

Figure B.1: $q(c)$ for different real values of c_0 and c when $\alpha \in \{0.5, 1, 2, 5, 10, 15\}$.

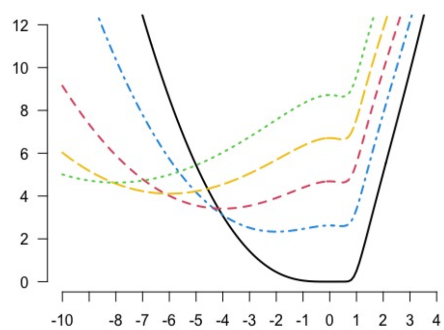
(a) $\alpha = 0.5$



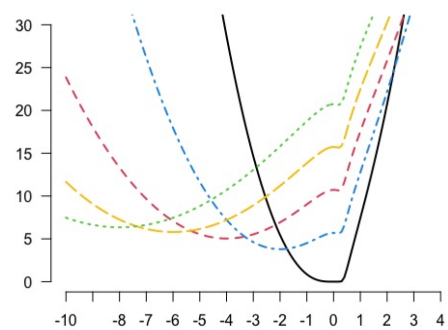
(b) $\alpha = 1$



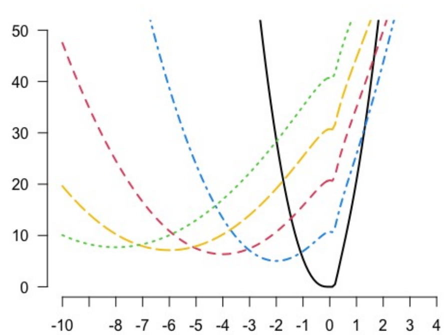
(c) $\alpha = 2$



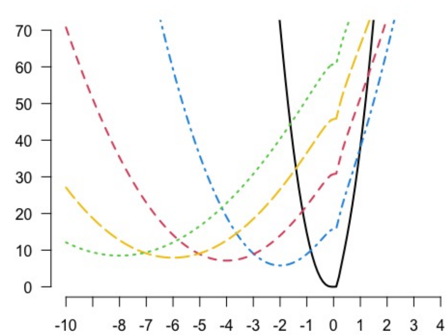
(d) $\alpha = 5$



(e) $\alpha = 10$



(f) $\alpha = 15$

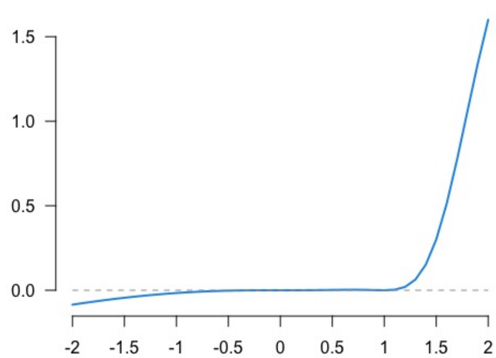


Note: The horizontal axis represents values of c. The black (solid), blue (dash-dotted), red (dashed), yellow (long dashed) and green (dotted) lines are for c_0 equal to 0, -2, -4, -6 and -8, respectively.

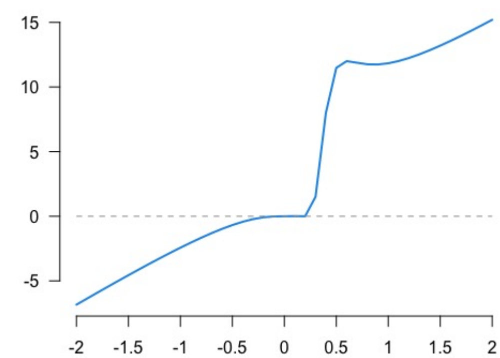
The problem when c_0 is close to zero remains irrespective of the value of α. Figure B.2 below depicts $dq(c)/dc$ for different values of c and α when $c_0 = 0$. As can be seen from (a)–(d), the values where $dq(c)/dc = 0$ cluster very closely around $c_0 = 0$. Of course, under $c_0 \neq 0$, the roots cluster for large values of α as well, because the graph becomes narrow. However, the global minimum is always distinct from the local stationary points. Overall, while the third root depends on α, it is not detrimental to the performance of FA as long as c_0 sufficiently far away from zero.

Figure B.2: $dq(c)/dc$ for $c_0 = 0$ and $\alpha \in \{1, 5, 10, 15\}$.

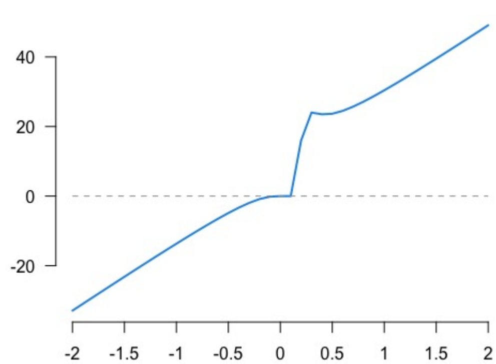
(a) $\alpha = 1$



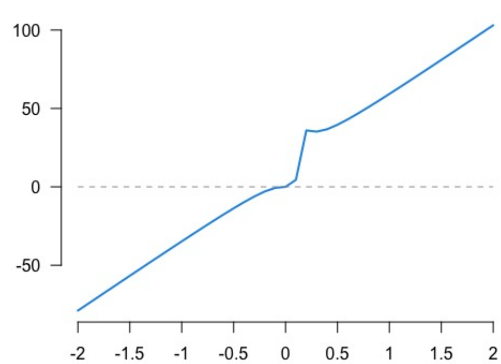
(b) $\alpha = 5$



(c) $\alpha = 10$



(d) $\alpha = 15$



C Derivation of ℓ^*

The derivations contained in this section and the next build heavily on Abadir and Magnus (2005). All the arguments used can be found in this book. As a matter of notation, A and B will be used to generic functions of x, with a and b denoting generic constants. We define the matrix derivative operator D_x, which is such that if the matrix function $F = F(x)$ is $m \times p$ and x is $n \times q$, then $\mathrm{D}_x F = \partial \mathrm{vec}\, F / \partial (\mathrm{vec}\, x)'$ is $mp \times nq$. Hence, denoting by d_x the matrix differential, then $\mathrm{d}_x \,\mathrm{vec}\, F = F \mathrm{d}_x \,\mathrm{vec}\, x$, or $\mathrm{D}_x F = \mathrm{d}_x \,\mathrm{vec}\, F / \mathrm{d}_x \,\mathrm{vec}\, x$.

The derivation of the stated expression for $\ell^*(\theta)$ starts by noting that

$$\ell^*(\theta) = -\frac{N}{2} Q^*(\theta_2), \tag{C.1}$$

where $\theta_2 = (\sigma^2, \rho)'$ and $Q^*(\theta_2) = \max_{S_\lambda} Q(\theta)$. Therefore, we derive $Q^*(\theta_2)$. Given that $Q(\theta) = \log(|\Sigma(\theta)|) + \mathrm{tr}(S_y \Sigma^{-1}(\theta))$, we start from the first term. In particular,

$$\begin{aligned} \log(|\Sigma(\theta)|) &= \log(|\sigma^2 \Gamma(\rho) \Lambda(S_\lambda, \sigma^2) \Gamma(\rho)'|) \\ &= \log((\sigma^2)^T |\Gamma(\rho)|^2 \times |\Lambda(S_\lambda, \sigma^2)|) = T \log(\sigma^2) + \log(|\Lambda(S_\lambda, \sigma^2)|), \end{aligned} \tag{C.2}$$

where the last equality follows from the fact that $|\Gamma(\rho)| = |\Gamma(\rho)'| = 1$ and $|aA| = a^n |A|$ for any $A \in \mathbb{R}^{n \times n}$. Moving on to the second term,

$$\begin{aligned} \mathrm{tr}(S_y \Sigma(\theta)^{-1}) &= \mathrm{tr}(S_y [\sigma^2 \Gamma(\rho) \Lambda(S_\lambda, \sigma^2) \Gamma(\rho)']^{-1}) \\ &= \mathrm{tr}(\sigma^{-2} S_y \Gamma(\rho)^{-1\prime} \Lambda(S_\lambda, \sigma^2)^{-1} \Gamma(\rho)^{-1}) = \sigma^{-2} \mathrm{tr}(\Gamma(\rho)^{-1} S_y \Gamma(\rho)^{-1\prime} \Lambda(S_\lambda, \sigma^2)^{-1}) \\ &= \sigma^{-2} \mathrm{tr}(G(\rho) \Lambda(S_\lambda, \sigma^2)^{-1}), \end{aligned} \tag{C.3}$$

which allows us to obtain

$$Q(\theta) = T \log(\sigma^2) + \log(|\Lambda(S_\lambda, \sigma^2)|) + \sigma^{-2} \mathrm{tr}(G(\rho) \Lambda(S_\lambda, \sigma^2)^{-1}). \tag{C.4}$$

Now, we will take the matrix derivative with respect to S_λ and we start from $\mathrm{D}_{S_\lambda} \Lambda(S_\lambda, \sigma^2)$. Specifically,

$$\begin{aligned} \mathrm{D}_{S_\lambda} \Lambda(S_\lambda, \sigma^2) &= \mathrm{D}_{S_\lambda} (I_T + \sigma^{-2} \Gamma(\rho)^{-1} D S_\lambda D' \Gamma(\rho)^{-1\prime}) \\ &= \sigma^{-2} \mathrm{D}_{S_\lambda} (\Gamma(\rho)^{-1} D S_\lambda D' \Gamma(\rho)^{-1\prime}) = \sigma^{-2} (\Gamma(\rho)^{-1} D \otimes \Gamma(\rho)^{-1} D), \end{aligned} \tag{C.5}$$

which directly follows from the fact that $\mathrm{vec}(\Gamma(\rho)^{-1} D S_\lambda D' \Gamma(\rho)^{-1\prime}) = (\Gamma(\rho)^{-1} D \otimes \Gamma(\rho)^{-1} D) \mathrm{vec}\, S_\lambda$ using $\mathrm{vec}\,(ABC) = (C' \otimes A) \mathrm{vec}\, B$ and the properties of matrix derivative operator. Now, using $\mathrm{D}_x \log(|A(x)|) = [\mathrm{vec}\,(A(x)^{-1\prime})]'\mathrm{D}_x A(x)$, we obtain

$$\begin{aligned}
&\mathrm{D}_{S_\lambda} \log(|\Lambda(S_\lambda,\sigma^2)|) = [\mathrm{vec}\,(\Lambda(S_\lambda,\sigma^2)^{-1\prime})]'\mathrm{D}_{S_\lambda}\Lambda(S_\lambda,\sigma^2) \\
&\quad = \sigma^{-2}[\mathrm{vec}\,(\Lambda(S_\lambda,\sigma^2)^{-1\prime})]'(\Gamma(\rho)^{-1}D \otimes \Gamma(\rho)^{-1}D). \qquad \text{(C.6)}
\end{aligned}$$

Next, we differentiate $\mathrm{tr}(G(\rho)\Lambda(S_\lambda,\sigma^2)^{-1})$. From $\mathrm{D}_x \mathrm{tr}(AB(x)^{-1}) = -[\mathrm{vec}\,(B(x)^{-1}AB(x)^{-1})']'\mathrm{D}_x B(x)$, where the matrix A is constant with respect to x, we obtain

$$\begin{aligned}
&\mathrm{D}_{S_\lambda} \mathrm{tr}(G(\rho)\Lambda(S_\lambda,\sigma^2)^{-1}) = -[\mathrm{vec}\,(\Lambda(S_\lambda,\sigma^2)^{-1}G(\rho)\Lambda(S_\lambda,\sigma^2)^{-1})']'\mathrm{D}_{S_\lambda}\Lambda(S_\lambda,\sigma^2) \\
&\quad = -\sigma^{-2}[\mathrm{vec}\,(\Lambda(S_\lambda,\sigma^2)^{-1}G(\rho)\Lambda(S_\lambda,\sigma^2)^{-1})']'(\Gamma(\rho)^{-1}D \otimes \Gamma(\rho)^{-1}D). \qquad \text{(C.7)}
\end{aligned}$$

Hence, we finally obtain

$$\begin{aligned}
&\mathrm{D}_{S_\lambda} Q(\theta) = \mathrm{D}_{S_\lambda} \log(|\Lambda(\hat{S}_\lambda,\sigma^2)|) + \sigma^{-2}\mathrm{D}_{S_\lambda}\mathrm{tr}(G(\rho)\Lambda(\hat{S}_\lambda,\sigma^2)^{-1}) \\
&\quad = \sigma^{-2}[\mathrm{vec}\,(\Lambda(\hat{S}_\lambda,\sigma^2)^{-1\prime})]'(\Gamma(\rho)^{-1}D \otimes \Gamma(\rho)^{-1}D) \\
&\quad - \sigma^{-4}[\mathrm{vec}\,(\Lambda(\hat{S}_\lambda,\sigma^2)^{-1}G(\rho)\Lambda(\hat{S}_\lambda,\sigma^2)^{-1})']'(\Gamma(\rho)^{-1}D \otimes \Gamma(\rho)^{-1}D) \\
&\quad = [\mathrm{vec}\,[\Lambda(\hat{S}_\lambda,\sigma^2)^{-1\prime} - \sigma^{-2}(\Lambda(\hat{S}_\lambda,\sigma^2)^{-1}G(\rho)\Lambda(\hat{S}_\lambda,\sigma^2)^{-1})']]'(\Gamma(\rho)^{-1}D \otimes \Gamma(\rho)^{-1}D) \\
&\quad = 0_{1\times m^2}, \qquad \text{(C.8)}
\end{aligned}$$

where the equality holds if and only if

$$G(\rho) = \sigma^2\Lambda(\hat{S}_\lambda,\sigma^2) = \sigma^2(I_T + \sigma^{-2}\Gamma(\rho)^{-1}D\hat{S}_\lambda D'\Gamma(\rho)^{-1\prime}). \qquad \text{(C.9)}$$

Hence,

$$\hat{S}_\lambda = \sigma^2(\Gamma(\rho)^{-1}D)^{+}(\sigma^{-2}G - I_T)(\Gamma(\rho)^{-1}D)^{+\prime}, \qquad \text{(C.10)}$$

where $A^{+} = (A'A)^{-1}A'$ is the Moon–Penrose inverse of A. Note that $\hat{S}_\lambda = \hat{S}_\lambda(\theta_2)$. Combining the results, the concentrated discrepancy function becomes

$$Q^*(\theta_2) = T\log(\sigma^2) + \log(|\hat{\Lambda}(\theta_2)|) + \sigma^{-2}\mathrm{tr}(G(\rho)\hat{\Lambda}(\theta_2)^{-1}) \qquad \text{(C.11)}$$

where $\hat{\Lambda}(\theta_2) = I_T + \sigma^2\Gamma(\rho)^{-1}D\hat{S}_\lambda D'\Gamma(\rho)^{-1\prime}$. The required expression for $\ell^*(\theta)$ is implied by this.

D Derivatives

In Lemma A we report first and second order derivatives of ℓ^* with respect to ρ. Before resenting them, we need to introduce some notation. We begin by defining $Q(\rho) =$

$D'\Gamma(\rho)^{-1\prime}\Gamma(\rho)^{-1}D$. This is a function of ρ. To simplify notation, however, in this appendix matrices such as $Q(\rho)$, $G(\rho)$, $L(\rho)$ and $\Sigma(\theta)$ will be written as Q, G, L and Σ, respectively, with the dependence on θ (and hence ρ) suppressed. The derivatives are stated in terms of the following scalars:

$$
\begin{aligned}
R_1 &= \operatorname{tr}(M_{\Gamma^{-1}D}GM_{\Gamma^{-1}D}L - \sigma^2 LM_{\Gamma^{-1}D}), \\
R_2 &= \operatorname{tr}[\sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D], \\
r_1 &= \operatorname{tr}(-\sigma^2 M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}(L'+L) + (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LLM_{\Gamma^{-1}D} \\
&- (G-\sigma^2 I_T)M_{\Gamma^{-1}D}(L'+L)M_{\Gamma^{-1}D}(L'+L)M_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}L(L'+L)M_{\Gamma^{-1}D}), \\
r_2 &= \sigma^2\operatorname{tr}[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(D'\Gamma^{-1\prime}(L'+L)G\Gamma^{-1}D + D'\Gamma^{-1\prime}G(L'+L)\Gamma^{-1}D) \\
&\times (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
&+ (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(D'\Gamma^{-1\prime}(L'+L)(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
&- D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
&- D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G(L'+L)\Gamma^{-1}D - 2D'\Gamma^{-1\prime}L'LM_{\Gamma^{-1}D}G\Gamma^{-1}D)].
\end{aligned}
$$

These scalars are all functions of θ_2. For ease of notation, however, we suppress this dependence on θ_2.

Lemma D.1. *For any vector D_t,*

(a) $\dfrac{1}{N}\dfrac{\partial \ell^*}{\partial \rho} = \sigma^{-2}(R_1 + R_2)$,

(b) $\dfrac{1}{N}\dfrac{\partial \ell^*}{\partial \sigma^2} = -\dfrac{T}{2\sigma^2} + \dfrac{m}{2\sigma^2} + \dfrac{1}{2\sigma^4}\operatorname{tr}(GM_{\Gamma^{-1}D})$,

(c) $\dfrac{1}{N}\dfrac{\partial^2 \ell^*}{(\partial \rho)^2} = \sigma^{-2}(r_1 + r_2)$,

(d) $\dfrac{1}{N}\dfrac{\partial^2 \ell^*}{(\partial \sigma^2)^2} = \dfrac{T}{2\sigma^4} - \dfrac{m}{2\sigma^4} - \sigma^{-6}\operatorname{tr}(GM_{\Gamma^{-1}D})$,

(e) $\dfrac{1}{N}\dfrac{\partial^2 \ell^*}{\partial \rho \partial \sigma^2} = -\sigma^{-4}\operatorname{tr}(M_{\Gamma^{-1}D}GM_{\Gamma^{-1}D}L)$.

Proof: Consider (a). We begin by recalling that

$$\ell^* = -\frac{NT}{2}\log(\sigma^2) - \frac{N}{2}\log(|\hat{\Lambda}|) - \frac{N}{2\sigma^2}\operatorname{tr}[G\hat{\Lambda}^{-1}], \tag{D.12}$$

implying

$$\frac{\partial \ell^*}{\partial \rho} = -\frac{N}{2}\mathrm{D}_\rho \log(|\hat{\Lambda}|) - \frac{N}{2\sigma^2}\mathrm{D}_\rho \operatorname{tr}(G\hat{\Lambda}^{-1}). \tag{D.13}$$

We need to evaluate two terms; $\mathrm{D}_\rho \log(|\hat{\Lambda}|)$ and $\mathrm{D}_\rho \operatorname{tr}(G\hat{\Lambda}^{-1})$. Consider $\mathrm{D}_\rho \log(|\hat{\Lambda}|)$. We can show that $\log(|\hat{\Lambda}|) = \log(|\sigma^{-2}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}|)$. Indeed, applying Sylvester's determinant identity, we obtain

$$\begin{aligned}
&|\hat{\Lambda}| = |I_T + \sigma^{-2}\Gamma^{-1}D\hat{S}_\lambda D'\Gamma^{-1\prime}| \\
&= |I_2 + \sigma^{-2}\hat{S}_\lambda D'\Gamma^{-1\prime}\Gamma^{-1}D| = |I_2 + \sigma^{-2}\hat{S}_\lambda Q|.
\end{aligned} \tag{D.14}$$

Because

$$\begin{aligned}
\hat{S}_\lambda &= \sigma^2(\Gamma^{-1}D)^+(\sigma^{-2}G - I_T)(\Gamma^{-1}D)^{+\prime} \\
&= (D'\Gamma^{-1\prime}\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}G\Gamma^{-1}D(D'\Gamma^{-1\prime}\Gamma^{-1}D)^{-1} - \sigma^2(D'\Gamma^{-1\prime}\Gamma^{-1}D)^{-1} \\
&= Q^{-1}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1} - \sigma^2 Q^{-1},
\end{aligned} \tag{D.15}$$

by the direct insertion we obtain

$$\begin{aligned}
&|\hat{\Lambda}| = |I_2 + \sigma^{-2}\hat{S}_\lambda Q| \\
&= |I_2 + \sigma^{-2}(Q^{-1}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1} - \sigma^2 Q^{-1})Q| = |\sigma^2 Q^{-1}D'\Gamma^{-1\prime}G\Gamma^{-1}D| \\
&= |\sigma^2 D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}|,
\end{aligned} \tag{D.16}$$

because the determinant of a transpose is the same. By using this result and $\mathrm{D}_x \log|A| = (\operatorname{vec}(A'^{-1}))'\mathrm{D}_x A$, we get

$$\begin{aligned}
\mathrm{D}_\rho \log(|\hat{\Lambda}|) &= \mathrm{D}_\rho \log(|\sigma^{-2}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}|) \\
&= [\operatorname{vec}(\sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}Q)]'\mathrm{D}_\rho(\sigma^{-2}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}),
\end{aligned} \tag{D.17}$$

where the last equality holds due to symmetry of Q and G and $(AB)^{-1\prime} = A^{-1\prime}B^{-1\prime}$. Consider $\sigma^{-2}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}$. From $\mathrm{d}_x AB = (\mathrm{d}_x A)B + A(\mathrm{d}_x B)$,

$$\begin{aligned}
&\mathrm{d}_\rho(\sigma^{-2}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}) \\
&= \sigma^{-2}D'\mathrm{d}_\rho(\Gamma^{-1\prime})G\Gamma^{-1}DQ^{-1} + \sigma^{-2}D'\Gamma^{-1\prime}(\mathrm{d}_\rho G)\Gamma^{-1}DQ^{-1} \\
&+ \sigma^{-2}D'\Gamma^{-1\prime}G(\mathrm{d}_\rho \Gamma^{-1})DQ^{-1} + \sigma^{-2}D'\Gamma^{-1\prime}G\Gamma^{-1}D(\mathrm{d}_\rho Q^{-1}).
\end{aligned}$$

Further use of $\operatorname{vec}(ABC) = (C' \otimes A)\operatorname{vec} B$ yields

$$\begin{aligned}
&\mathrm{D}_\rho(\sigma^{-2}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}) \\
&= \sigma^{-2}(Q^{-1}D'\Gamma^{-1\prime}G \otimes D')\mathrm{D}_\rho \Gamma^{-1\prime} + \sigma^{-2}(Q^{-1}D'\Gamma^{-1\prime} \otimes D'\Gamma^{-1\prime})\mathrm{D}_\rho G \\
&+ \sigma^{-2}(Q^{-1}D' \otimes D'\Gamma^{-1\prime}G)\mathrm{D}_\rho \Gamma^{-1} + \sigma^{-2}(I_T \otimes D'\Gamma^{-1\prime}G\Gamma^{-1}D)\mathrm{D}_\rho Q^{-1}.
\end{aligned} \tag{D.18}$$

In order to evaluate this expression, we need $\mathrm{D}_\rho\,\Gamma^{-1}$, $\mathrm{D}_\rho\,\Gamma^{-1\prime}$, $\mathrm{D}_\rho\,G$ and $\mathrm{D}_\rho\,Q^{-1}$. Consider $\mathrm{D}_\rho\,\Gamma^{-1}$. Here

$$\mathrm{d}_\rho\,\Gamma^{-1} = \mathrm{d}_\rho \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\rho & 1 & 0 & \dots & 0 \\ 0 & -\rho & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -\rho & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 0 \end{bmatrix} \mathrm{d}\rho = -J\mathrm{d}\rho.$$

Repeated use of $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\,B$ yields $\mathrm{D}_\rho\,\Gamma^{-1} = -(I_T \otimes I_T)\mathrm{vec}\,J$. Since $\mathrm{d}_x\,(A') = (\mathrm{d}_x\,A)'$, we also have $\mathrm{D}_\rho\,\Gamma^{-1\prime} = -(I_T \otimes I_T)\mathrm{vec}\,J'$. Moreover, from $\mathrm{d}_x\,AB = (\mathrm{d}_x\,A)B + A(\mathrm{d}_x\,B)$, we get

$$\mathrm{d}_\rho\,G = \mathrm{d}_\rho\,(\Gamma^{-1}S_y\Gamma^{-1\prime}) = (\mathrm{d}_\rho\,\Gamma^{-1})S_y\Gamma^{-1\prime} + \Gamma^{-1}S_y(\mathrm{d}_\rho\,\Gamma^{-1})',$$

and so, via $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\,B$,

$$\mathrm{vec}\,\mathrm{d}_\rho\,G = (\Gamma^{-1} \otimes \mathrm{d}_\rho\,\Gamma^{-1} + \mathrm{d}_\rho\,\Gamma^{-1} \otimes \Gamma^{-1})\mathrm{vec}\,S_y = -(\Gamma^{-1} \otimes J\mathrm{d}\rho + J\mathrm{d}\rho \otimes \Gamma^{-1})\mathrm{vec}\,S_y.$$

It follows that

$$\mathrm{D}_\rho\,G = -(\Gamma^{-1} \otimes J + J \otimes \Gamma^{-1})\mathrm{vec}\,S_y. \tag{D.19}$$

Next, consider $\mathrm{D}_\rho\,Q^{-1}$. From $\mathrm{d}_x\,A^{-1} = -A^{-1}(\mathrm{d}_x\,A)A^{-1}$, $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\,B$, and the symmetry of Q,

$$\mathrm{D}_\rho\,Q^{-1} = -(Q^{-1} \otimes Q^{-1})\mathrm{D}_\rho\,Q,$$

where

$$\mathrm{d}_\rho\,Q = D'[(\mathrm{d}_\rho\,\Gamma^{-1\prime})\Gamma^{-1} + \Gamma^{-1\prime}(\mathrm{d}_\rho\,\Gamma^{-1})]D.$$

By using this, $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\,B$, $\mathrm{d}_\rho\,(A') = (\mathrm{d}_\rho\,A)'$, $\mathrm{D}_\rho\,\Gamma^{-1} = -(I_T \otimes I_T)\mathrm{vec}\,J$ and $\mathrm{D}_\rho\,\Gamma^{-1\prime} = -(I_T \otimes I_T)\mathrm{vec}\,J'$, we can show that

$$\begin{aligned} \mathrm{D}_\rho\,Q &= (D'\Gamma^{-1\prime} \otimes D')\mathrm{D}_\rho\,\Gamma^{-1\prime} + (D' \otimes D'\Gamma^{-1\prime})\mathrm{D}_\rho\,\Gamma^{-1} \\ &= -(D'\Gamma^{-1\prime} \otimes D')\mathrm{vec}\,J' - (D' \otimes D'\Gamma^{-1\prime})\mathrm{vec}\,J. \end{aligned} \tag{D.20}$$

Hence, since $(A \otimes B)(C \otimes D) = AC \otimes BD$,

$$\begin{aligned} \mathrm{D}_\rho\,Q^{-1} &= (Q^{-1} \otimes Q^{-1})[(D'\Gamma^{-1\prime} \otimes D')\mathrm{vec}\,J' + (D' \otimes D'\Gamma^{-1\prime})\mathrm{vec}\,J] \\ &= (Q^{-1}D'\Gamma^{-1\prime} \otimes Q^{-1}D')\mathrm{vec}\,J' + (Q^{-1}D' \otimes Q^{-1}D'\Gamma^{-1\prime})\mathrm{vec}\,J. \end{aligned}$$

By inserting the expressions for $\mathrm{D}_\rho\,\Gamma^{-1}$, $\mathrm{D}_\rho\,\Gamma^{-1\prime}$, $\mathrm{D}_\rho\,G$ and $\mathrm{D}_\rho\,Q^{-1}$ into (D.18) and using $(A\otimes B)(C\otimes D) = AC\otimes BD$, we obtain

$$\begin{aligned}
&\mathrm{D}_\rho\,(\sigma^{-2}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}) \\
&= \; -\sigma^{-2}(Q^{-1}D'\Gamma^{-1\prime}G\otimes D')\mathrm{vec}\,J' - \sigma^{-2}(Q^{-1}D'\otimes D'\Gamma^{-1\prime}G)\mathrm{vec}\,J \\
&- \; \sigma^{-2}(Q^{-1}D'\Gamma^{-1\prime}\Gamma^{-1}\otimes D'\Gamma^{-1\prime}J)\mathrm{vec}\,S_y - \sigma^{-2}(Q^{-1}D'\Gamma^{-1\prime}J\otimes D'\Gamma^{-1\prime}\Gamma^{-1})\mathrm{vec}\,S_y \\
&+ \; \sigma^{-2}(Q^{-1}D'\Gamma^{-1\prime}\otimes D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D')\mathrm{vec}\,J' \\
&+ \; \sigma^{-2}(Q^{-1}D'\otimes D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime})\mathrm{vec}\,J.
\end{aligned} \tag{D.21}$$

Further use of $\mathrm{vec}(ABC) = (C'\otimes A)\mathrm{vec}\,B$ gives

$$\begin{aligned}
&\mathrm{D}_\rho\,(\sigma^{-2}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}) \\
&= \; -\sigma^{-2}\mathrm{vec}\,(D'J'G\Gamma^{-1}DQ^{-1}) - \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}GJDQ^{-1}) \\
&- \; \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}JS_y\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}) - \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}\Gamma^{-1}S_yJ'\Gamma^{-1}DQ^{-1}) \\
&+ \; \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'J'\Gamma^{-1}DQ^{-1}) \\
&+ \; \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}JDQ^{-1}) \\
&= \; -\sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}\Gamma'J'G\Gamma^{-1}DQ^{-1}) - \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}GJ\Gamma\Gamma^{-1}DQ^{-1}) \\
&- \; \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}J\Gamma S_u\Gamma'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}) - \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma S_u\Gamma'J'\Gamma^{-1}DQ^{-1}) \\
&+ \; \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}\Gamma'J'\Gamma^{-1}DQ^{-1}) \\
&+ \; \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}J\Gamma\Gamma^{-1}DQ^{-1}) \\
&= \; -\sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}L'G\Gamma^{-1}DQ^{-1}) - \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}GL\Gamma^{-1}DQ^{-1}) \\
&- \; \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}LG\Gamma^{-1}DQ^{-1}) - \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}GL'\Gamma^{-1}DQ^{-1}) \\
&+ \; \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}L'\Gamma^{-1}DQ^{-1}) \\
&+ \; \sigma^{-2}\mathrm{vec}\,(D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}L\Gamma^{-1}DQ^{-1}),
\end{aligned} \tag{D.22}$$

where the second equality is due to $S_y = \Gamma S_u\Gamma'$, while the last equality holds because of $J\Gamma = L$ and $S_u = G$. Since $\mathrm{tr}\,(A'B) = (\mathrm{vec}\,A)'\mathrm{vec}\,B$, this implies that $\mathrm{D}_\rho\,\log(|\hat{\Lambda}|)$ can be written in the following fashion:

$$
\begin{aligned}
&\mathrm{D}_\rho \log(|\hat{\Lambda}|) \\
&= [\mathrm{vec}\,(\sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}Q)]'\mathrm{D}_\rho\,(\sigma^{-2}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}), \\
&= [\mathrm{vec}\,(\sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}Q)]'\sigma^{-2}\mathrm{vec}\,(-D'\Gamma^{-1\prime}L'G\Gamma^{-1}DQ^{-1} \\
&- D'\Gamma^{-1\prime}GL\Gamma^{-1}DQ^{-1} - D'\Gamma^{-1\prime}LG\Gamma^{-1}DQ^{-1} - D'\Gamma^{-1\prime}GL'\Gamma^{-1}DQ^{-1} \\
&+ D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}L'\Gamma^{-1}DQ^{-1} + D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}L\Gamma^{-1}DQ^{-1}) \\
&= \mathrm{tr}\,[Q(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(-D'\Gamma^{-1\prime}L'G\Gamma^{-1}DQ^{-1} \\
&- D'\Gamma^{-1\prime}GL\Gamma^{-1}DQ^{-1} - D'\Gamma^{-1\prime}LG\Gamma^{-1}DQ^{-1} - D'\Gamma^{-1\prime}GL'\Gamma^{-1}DQ^{-1} \\
&+ D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}L'\Gamma^{-1}DQ^{-1} + D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}L\Gamma^{-1}DQ^{-1})] \\
&= \mathrm{tr}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(-2D'\Gamma^{-1\prime}(L'+L)G\Gamma^{-1}D \\
&+ D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}(L'+L)\Gamma^{-1}D)]. \qquad \text{(D.23)}
\end{aligned}
$$

This can be simplified. We begin by noting that if we let $P_{\Gamma^{-1}D} = \Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}$ and $M_{\Gamma^{-1}D} = I_T - P_{\Gamma^{-1}D}$, then

$$
\begin{aligned}
&\mathrm{D}_\rho \log(|\hat{\Lambda}|) \\
&= \mathrm{tr}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(-2D'\Gamma^{-1\prime}(L'+L)G\Gamma^{-1}D \\
&+ D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}(L'+L)\Gamma^{-1}D)] \\
&= \mathrm{tr}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(-2D'\Gamma^{-1\prime}(L'+L)G\Gamma^{-1}D + 2D'\Gamma^{-1\prime}GP_{\Gamma^{-1}D}(L'+L)\Gamma^{-1}D \\
&- D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}(L'+L)\Gamma^{-1}D)] \\
&= -2\mathrm{tr}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L'+L)(I_T - P_{\Gamma^{-1}D})G\Gamma^{-1}D] \\
&- \mathrm{tr}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}(L'+L)\Gamma^{-1}D)] \\
&= -2\mathrm{tr}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D] \\
&- \mathrm{tr}\,[Q^{-1}D'\Gamma^{-1\prime}(L'+L)\Gamma^{-1}D] \\
&= -2\mathrm{tr}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D] - 2\mathrm{tr}\,[LP_{\Gamma^{-1}D}] \\
&= -2\mathrm{tr}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D] + 2\mathrm{tr}\,[LM_{\Gamma^{-1}D}], \qquad \text{(D.24)}
\end{aligned}
$$

where the last equality holds because $\mathrm{tr}\,L = 0$.

Let us now consider the second term in (D.13), $\mathrm{D}_\rho\,\mathrm{tr}\,(G\hat{\Lambda}^{-1})$. We begin by inserting $\hat{\Lambda}^{-1} = I_T - \Gamma^{-1}DKD'\Gamma^{-1\prime}$ and $K = Q^{-1} - \sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}$, which we obtain from The Woodbury identity stating that $(A + CBC')^{-1} = A^{-1} - A^{-1}C(B^{-1} + C'A^{-1}C)^{-1}C'A^{-1}$ (see

AAbadir and Magnus, 2005, Exercise 5.17). Application of this identity to $\hat{\Lambda}^{-1}$ yields, with $K = (\sigma^2 \hat{S}_\lambda^{-1} + Q)^{-1}$,

$$\hat{\Lambda}^{-1} = I_T - \Gamma^{-1} D(\sigma^2 \hat{S}_\lambda^{-1} + Q)^{-1} D'\Gamma^{-1\prime} = I_T - \Gamma^{-1} DKD'\Gamma^{-1\prime}. \tag{D.25}$$

Using the expression for $\hat{S}_\lambda$ again and $(A + CBC')^{-1} = A^{-1} - A^{-1}C(B^{-1} + C'A^{-1}C)^{-1}C'A^{-1}$,

$$\begin{aligned} K &= (\sigma^2 \hat{S}_\lambda^{-1} + Q)^{-1} = Q^{-1} - Q^{-1}(\sigma^{-2}\hat{S}_\lambda + Q^{-1})^{-1}Q^{-1} \\ &= Q^{-1} - \sigma^2 (D'\Gamma^{-1\prime} G \Gamma^{-1} D)^{-1}. \end{aligned} \tag{D.26}$$

The direct insertion then leads to

$$\begin{aligned} &\operatorname{tr}(G\hat{\Lambda}^{-1}) \\ &= \operatorname{tr}[G(I_T - \Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime} + \sigma^2\Gamma^{-1}D(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime})] \\ &= \operatorname{tr}[G(I_T - P_{\Gamma^{-1}D})] + \sigma^2 \operatorname{tr}[G\Gamma^{-1}D(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}] \\ &= \operatorname{tr}(GM_{\Gamma^{-1}D}) + \sigma^2 \operatorname{tr}[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}G\Gamma^{-1}D] \\ &= \operatorname{tr}(GM_{\Gamma^{-1}D}) + \sigma^2 \operatorname{tr} I_m = \operatorname{tr}(GM_{\Gamma^{-1}D}) + \sigma^2 m, \end{aligned} \tag{D.27}$$

implying that

$$\mathrm{D}_\rho \operatorname{tr}(G\hat{\Lambda}^{-1}) = \mathrm{D}_\rho \operatorname{tr}(GM_{\Gamma^{-1}D}) = (\operatorname{vec} I_T)'\mathrm{D}_\rho (GM_{\Gamma^{-1}D}). \tag{D.28}$$

By using $\mathrm{d}_x AB = (\mathrm{d}_x A)B + A(\mathrm{d}_x B)$, we get

$$\mathrm{d}_\rho (GM_{\Gamma^{-1}D}) = (\mathrm{d}_\rho G)M_{\Gamma^{-1}D} + G(\mathrm{d}_\rho M_{\Gamma^{-1}D}) = (\mathrm{d}_\rho G)M_{\Gamma^{-1}D} - G(\mathrm{d}_\rho P_{\Gamma^{-1}D}),$$

and so, via $\operatorname{vec}(ABC) = (C' \otimes A)\operatorname{vec} B$,

$$\mathrm{D}_\rho (GM_{\Gamma^{-1}D}) = (M_{\Gamma^{-1}D} \otimes I_T)\mathrm{D}_\rho G - (I_T \otimes G)\mathrm{D}_\rho P_{\Gamma^{-1}D}.$$

Consider $\mathrm{D}_\rho P_{\Gamma^{-1}D}$. Repeated use of $\mathrm{d}_x AB = (\mathrm{d}_x A)B + A(\mathrm{d}_x B)$ yields

$$\begin{aligned} \mathrm{d}_\rho P_{\Gamma^{-1}D} &= \mathrm{d}_\rho (\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}) \\ &= (\mathrm{d}_\rho \Gamma^{-1})DQ^{-1}D'\Gamma^{-1\prime} + \Gamma^{-1}D(\mathrm{d}_\rho Q^{-1})D'\Gamma^{-1\prime} + \Gamma^{-1}DQ^{-1}D'(\mathrm{d}_\rho \Gamma^{-1\prime}), \end{aligned}$$

and by further use $\operatorname{vec}(ABC) = (C' \otimes A)\operatorname{vec} B$, $(A \otimes B)(C \otimes D) = AC \otimes BD$ and $A \otimes B +$

$A \otimes C = A \otimes (B + C)$, we get

$$
\begin{aligned}
& \mathrm{D}_\rho \, P_{\Gamma^{-1}D} \\
&\quad = (\Gamma^{-1}DQ^{-1}D' \otimes I_T)\mathrm{D}_\rho \, \Gamma^{-1} + (I_T \otimes \Gamma^{-1}DQ^{-1}D')\mathrm{D}_\rho \, \Gamma^{-1\prime} \\
&\quad + (\Gamma^{-1}D \otimes \Gamma^{-1}D)\mathrm{D}_\rho \, Q^{-1} \\
&\quad = -(\Gamma^{-1}DQ^{-1}D' \otimes I_T)\mathrm{vec}\, J - (I_T \otimes \Gamma^{-1}DQ^{-1}D')\mathrm{vec}\, J' \\
&\quad + (\Gamma^{-1}D \otimes \Gamma^{-1}D)(Q^{-1}D'\Gamma^{-1\prime} \otimes Q^{-1}D')\mathrm{vec}\, J' \\
&\quad + (\Gamma^{-1}D \otimes \Gamma^{-1}D)(Q^{-1}D' \otimes Q^{-1}D'\Gamma^{-1\prime})\mathrm{vec}\, J \\
&\quad = -(\Gamma^{-1}DQ^{-1}D' \otimes I_T)\mathrm{vec}\, J - (I_T \otimes \Gamma^{-1}DQ^{-1}D')\mathrm{vec}\, J' \\
&\quad + (\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime} \otimes \Gamma^{-1}DQ^{-1}D')\mathrm{vec}\, J' \\
&\quad + (\Gamma^{-1}DQ^{-1}D' \otimes \Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime})\mathrm{vec}\, J \\
&\quad = -(\Gamma^{-1}DQ^{-1}D' \otimes I_T)\mathrm{vec}\, J - (I_T \otimes \Gamma^{-1}DQ^{-1}D')\mathrm{vec}\, J' \\
&\quad + (P_{\Gamma^{-1}D} \otimes \Gamma^{-1}DQ^{-1}D')\mathrm{vec}\, J' + (\Gamma^{-1}DQ^{-1}D' \otimes P_{\Gamma^{-1}D})\mathrm{vec}\, J \\
&\quad = -(\Gamma^{-1}DQ^{-1}D' \otimes I_T)\mathrm{vec}\, J - (I_T \otimes \Gamma^{-1}DQ^{-1}D')\mathrm{vec}\, J' \\
&\quad + (P_{\Gamma^{-1}D} \otimes \Gamma^{-1}DQ^{-1}D')\mathrm{vec}\, J' + (\Gamma^{-1}DQ^{-1}D' \otimes P_{\Gamma^{-1}D})\mathrm{vec}\, J \\
&\quad = -(\Gamma^{-1}DQ^{-1}D' \otimes (I_T - P_{\Gamma^{-1}D}))\mathrm{vec}\, J - ((I_T - P_{\Gamma^{-1}D}) \otimes \Gamma^{-1}DQ^{-1}D')\mathrm{vec}\, J' \\
&\quad = -(\Gamma^{-1}DQ^{-1}D' \otimes M_{\Gamma^{-1}D})\mathrm{vec}\, J - (M_{\Gamma^{-1}D} \otimes \Gamma^{-1}DQ^{-1}D')\mathrm{vec}\, J' \\
&\quad = -(\Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}\Gamma' \otimes M_{\Gamma^{-1}D})\mathrm{vec}\, J - (M_{\Gamma^{-1}D} \otimes \Gamma^{-1}DQ^{-1}D'\Gamma^{-1\prime}\Gamma')\mathrm{vec}\, J' \\
&\quad = -(P_{\Gamma^{-1}D}\Gamma' \otimes M_{\Gamma^{-1}D})\mathrm{vec}\, J - (M_{\Gamma^{-1}D} \otimes P_{\Gamma^{-1}D}\Gamma')\mathrm{vec}\, J'.
\end{aligned}
$$

By inserting the expressions for $\mathrm{D}_\rho \, G$ and $\mathrm{D}_\rho \, P_{\Gamma^{-1}D}$ into (D.29) and simplifying, we obtain the following expression for $\mathrm{D}_\rho \, (GM_{\Gamma^{-1}D})$:

$$
\begin{aligned}
& \mathrm{D}_\rho \, (GM_{\Gamma^{-1}D}) \\
&\quad = (M_{\Gamma^{-1}D} \otimes I_T)\mathrm{D}_\rho \, G - (I_T \otimes G)\mathrm{D}_\rho \, P_{\Gamma^{-1}D} \\
&\quad = -(M_{\Gamma^{-1}D} \otimes I_T)(\Gamma^{-1} \otimes J)\mathrm{vec}\, S_y - (M_{\Gamma^{-1}D} \otimes I_T)(J \otimes \Gamma^{-1})\mathrm{vec}\, S_y \\
&\quad + (I_T \otimes G)(P_{\Gamma^{-1}D}\Gamma' \otimes M_{\Gamma^{-1}D})\mathrm{vec}\, J + (I_T \otimes G)(M_{\Gamma^{-1}D} \otimes P_{\Gamma^{-1}D}\Gamma')\mathrm{vec}\, J' \\
&\quad = -(M_{\Gamma^{-1}D}\Gamma^{-1} \otimes J)\mathrm{vec}\, S_y - (M_{\Gamma^{-1}D}J \otimes \Gamma^{-1})\mathrm{vec}\, S_y \\
&\quad + (P_{\Gamma^{-1}D}\Gamma' \otimes GM_{\Gamma^{-1}D})\mathrm{vec}\, J + (M_{\Gamma^{-1}D} \otimes GP_{\Gamma^{-1}D}\Gamma')\mathrm{vec}\, J',
\end{aligned}
$$

which can be simplified using $\text{vec}(ABC) = (C' \otimes A)\text{vec}\, B$, $S_y = \Gamma S_u \Gamma'$, $J\Gamma = L$ and $S_u = G$;

$$\begin{aligned}
&\mathrm{D}_\rho\,(GM_{\Gamma^{-1}D}) \\
&= -(M_{\Gamma^{-1}D}\Gamma^{-1} \otimes J)\text{vec}\, S_y - (M_{\Gamma^{-1}D}J \otimes \Gamma^{-1})\text{vec}\, S_y \\
&+ (P_{\Gamma^{-1}D}\Gamma' \otimes GM_{\Gamma^{-1}D})\text{vec}\, J + (M_{\Gamma^{-1}D} \otimes GP_{\Gamma^{-1}D}\Gamma')\text{vec}\, J' \\
&= -\text{vec}\,(J\Gamma S_u \Gamma' \Gamma^{-1\prime} M_{\Gamma^{-1}D}) - \text{vec}\,(\Gamma^{-1}\Gamma S_u \Gamma' J' M_{\Gamma^{-1}D}) \\
&+ \text{vec}\,(GM_{\Gamma^{-1}D}J\Gamma P_{\Gamma^{-1}D}) + \text{vec}\,(GP_{\Gamma^{-1}D}\Gamma' J' M_{\Gamma^{-1}D}) \\
&= \text{vec}\,(-LGM_{\Gamma^{-1}D} - GL'M_{\Gamma^{-1}D} + GM_{\Gamma^{-1}D}LP_{\Gamma^{-1}D} + GP_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}).
\end{aligned}$$

Direct insertion into the expression for $\mathrm{D}_\rho\, \text{tr}\,(G\hat{\Lambda}^{-1})$ yields

$$\begin{aligned}
&\mathrm{D}_\rho\, \text{tr}\,(G\hat{\Lambda}^{-1}) \\
&= (\text{vec}\, I_T)'\mathrm{D}_\rho\,(GM_{\Gamma^{-1}D}) \\
&= (\text{vec}\, I_T)'\text{vec}\,(-LGM_{\Gamma^{-1}D} - GL'M_{\Gamma^{-1}D} + GM_{\Gamma^{-1}D}LP_{\Gamma^{-1}D} + GP_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}) \\
&= \text{tr}\,(-LGM_{\Gamma^{-1}D} - GL'M_{\Gamma^{-1}D} + GM_{\Gamma^{-1}D}LP_{\Gamma^{-1}D} + GP_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}) \\
&= 2\text{tr}\,(-GM_{\Gamma^{-1}D}L + GM_{\Gamma^{-1}D}LP_{\Gamma^{-1}D}) = 2\text{tr}\,[-GM_{\Gamma^{-1}D}L(I_T - P_{\Gamma^{-1}D})] \\
&= -2\text{tr}\,(GM_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}) \qquad (D.29)
\end{aligned}$$

The above results for $\mathrm{D}_\rho\ \log(|\hat{\Lambda}|)$ and $\mathrm{D}_\rho\, \text{tr}\,(G\hat{\Lambda}^{-1})$ lead to the following expression for $\partial\ell^*/\partial\rho$:

$$\begin{aligned}
\frac{2\sigma^2}{N}\frac{\partial\ell^*}{\partial\rho} &= -\sigma^2\mathrm{D}_\rho\ \log(|\hat{\Lambda}|) - \mathrm{D}_\rho\, \text{tr}\,(G\hat{\Lambda}^{-1}) \\
&= 2\text{tr}\,[\sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L' + L)GM_{\Gamma^{-1}D}\Gamma^{-1}D] \\
&- 2\sigma^2\text{tr}\,(LM_{\Gamma^{-1}D}) + 2\text{tr}\,(M_{\Gamma^{-1}D}GM_{\Gamma^{-1}D}L) \\
&= 2(R_1 + R_2), \qquad (D.30)
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= \text{tr}\,(M_{\Gamma^{-1}D}GM_{\Gamma^{-1}D}L - \sigma^2 LM_{\Gamma^{-1}D}), \\
R_2 &= \text{tr}\,[\sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L' + L)M_{\Gamma^{-1}D}G\Gamma^{-1}D].
\end{aligned}$$

This establishes (a).

Let us now consider (b). The starting point is

$$\frac{2\sigma^2}{N}\frac{\partial^2\ell^*}{(\partial\rho)^2} = 2(\mathrm{D}_\rho R_1 + \mathrm{D}_\rho R_2).$$

We begin with $\mathrm{D}_\rho R_1$. From definition of R_1, we have

$$\begin{aligned} R_1 &= \mathrm{tr}\,(M_{\Gamma^{-1}D} G M_{\Gamma^{-1}D} L - \sigma^2 L M_{\Gamma^{-1}D}) \\ &= \mathrm{tr}\,(G M_{\Gamma^{-1}D} L M_{\Gamma^{-1}D} - \sigma^2 M_{\Gamma^{-1}D} L M_{\Gamma^{-1}D}) \\ &= \mathrm{tr}\,[(G - \sigma^2 I_T) M_{\Gamma^{-1}D} L M_{\Gamma^{-1}D}]. \end{aligned}$$

Hence, via $\mathrm{tr}\,(A'B) = (\mathrm{vec}\,A)'\mathrm{vec}\,B$,

$$\begin{aligned} \mathrm{D}_\rho R_1 &= \mathrm{D}_\rho \mathrm{tr}\,[(G - \sigma^2 I_T) M_{\Gamma^{-1}D} L M_{\Gamma^{-1}D}] \\ &= (\mathrm{vec}\,I_T)' \mathrm{D}_\rho [(G - \sigma^2 I_T) M_{\Gamma^{-1}D} L M_{\Gamma^{-1}D}]. \end{aligned}$$

By use of $\mathrm{d}_x\,AB = (\mathrm{d}_x\,A)B + A(\mathrm{d}_x\,B)$, it is clear that

$$\begin{aligned} & \mathrm{d}_\rho [(G - \sigma^2 I_T) M_{\Gamma^{-1}D} L M_{\Gamma^{-1}D}] \\ &= [\mathrm{d}_\rho (G - \sigma^2 I_T)] M_{\Gamma^{-1}D} L M_{\Gamma^{-1}D} + (G - \sigma^2 I_T)(\mathrm{d}_\rho M_{\Gamma^{-1}D}) L M_{\Gamma^{-1}D} \\ &+ (G - \sigma^2 I_T) M_{\Gamma^{-1}D} (\mathrm{d}_\rho L) M_{\Gamma^{-1}D} + (G - \sigma^2 I_T) M_{\Gamma^{-1}D} L (\mathrm{d}_\rho M_{\Gamma^{-1}D}) \\ &= (\mathrm{d}_\rho G) M_{\Gamma^{-1}D} L M_{\Gamma^{-1}D} - (G - \sigma^2 I_T)(\mathrm{d}_\rho P_{\Gamma^{-1}D}) L M_{\Gamma^{-1}D} \\ &+ (G - \sigma^2 I_T) M_{\Gamma^{-1}D} (\mathrm{d}_\rho L) M_{\Gamma^{-1}D} - (G - \sigma^2 I_T) M_{\Gamma^{-1}D} L (\mathrm{d}_\rho P_{\Gamma^{-1}D}), \end{aligned}$$

where the last equality holds due to $\mathrm{d}_\rho(G - \sigma^2 I_T) = \mathrm{d}_\rho G$ and $\mathrm{d}_\rho M_{\Gamma^{-1}D} = \mathrm{d}_\rho(I_T - P_{\Gamma^{-1}D}) = -\mathrm{d}_\rho P_{\Gamma^{-1}D}$. By using this and $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\,B$, we obtain

$$\begin{aligned} & \mathrm{D}_\rho [(G - \sigma^2 I_T) M_{\Gamma^{-1}D} L M_{\Gamma^{-1}D}] \\ &= (M_{\Gamma^{-1}D} L' M_{\Gamma^{-1}D} \otimes I_T) \mathrm{D}_\rho G - (M_{\Gamma^{-1}D} L' \otimes (G - \sigma^2 I_T)) \mathrm{D}_\rho P_{\Gamma^{-1}D} \\ &+ (M_{\Gamma^{-1}D} \otimes (G - \sigma^2 I_T) M_{\Gamma^{-1}D}) \mathrm{D}_\rho L - (I_T \otimes (G - \sigma^2 I_T) M_{\Gamma^{-1}D} L) \mathrm{D}_\rho P_{\Gamma^{-1}D}, \end{aligned} \tag{D.31}$$

where $\mathrm{D}_\rho G$ and $\mathrm{D}_\rho P_{\Gamma^{-1}D}$ are known from before. As for $\mathrm{D}_\rho L$, we use $L = J\Gamma$ and, by applying $\mathrm{d}_x\,AB = (\mathrm{d}_x\,A)B + A(\mathrm{d}_x\,B)$, we get $\mathrm{d}_\rho L = J(\mathrm{d}_\rho \Gamma)$, where

$$\mathrm{d}_\rho\,\Gamma = \mathrm{d}_\rho \begin{bmatrix} 1 & 0 & \dots & 0 \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \rho^{T-1} & \dots & \rho & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 2\rho & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ (T-1)\rho^{T-2} & \dots & 2\rho & 1 & 0 \end{bmatrix} \mathrm{d}\rho = \Gamma J \Gamma \mathrm{d}\rho.$$

By using this, $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\,B$ and $L = J\Gamma$, we obtain

$$\mathrm{D}_\rho\,L = \mathrm{vec}\,(J\Gamma J\Gamma) = (\Gamma' \otimes J\Gamma)\mathrm{vec}\,J = (\Gamma' \otimes L)\mathrm{vec}\,J. \tag{D.32}$$

Insertion into (D.31) yields

$$
\begin{aligned}
& \mathrm{D}_\rho[(G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}] \\
= \;& -(M_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}\otimes I_T)(\Gamma^{-1}\otimes J)\mathrm{vec}\, S_y - (M_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}\otimes I_T)(J\otimes\Gamma^{-1})\mathrm{vec}\, S_y \\
+ \;& (M_{\Gamma^{-1}D}L'\otimes(G-\sigma^2 I_T))(P_{\Gamma^{-1}D}\Gamma'\otimes M_{\Gamma^{-1}D})\mathrm{vec}\, J \\
+ \;& (M_{\Gamma^{-1}D}L'\otimes(G-\sigma^2 I_T))(M_{\Gamma^{-1}D}\otimes P_{\Gamma^{-1}D}\Gamma')\mathrm{vec}\, J' \\
+ \;& (I_T\otimes(G-\sigma^2 I_T)M_{\Gamma^{-1}D}L)(P_{\Gamma^{-1}D}\Gamma'\otimes M_{\Gamma^{-1}D})\mathrm{vec}\, J \\
+ \;& (I_T\otimes(G-\sigma^2 I_T)M_{\Gamma^{-1}D}L)(M_{\Gamma^{-1}D}\otimes P_{\Gamma^{-1}D}\Gamma')\mathrm{vec}\, J' \\
+ \;& (M_{\Gamma^{-1}D}\otimes(G-\sigma^2 I_T)M_{\Gamma^{-1}D})(\Gamma'\otimes L)\mathrm{vec}\, J \\
= \;& -(M_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}\Gamma^{-1}\otimes J)\mathrm{vec}\, S_y - (M_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}J\otimes\Gamma^{-1})\mathrm{vec}\, S_y \\
+ \;& (M_{\Gamma^{-1}D}L'P_{\Gamma^{-1}D}\Gamma'\otimes(G-\sigma^2 I_T)M_{\Gamma^{-1}D})\mathrm{vec}\, J \\
+ \;& (M_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}\otimes(G-\sigma^2 I_T)P_{\Gamma^{-1}D}\Gamma')\mathrm{vec}\, J' \\
+ \;& (P_{\Gamma^{-1}D}\Gamma'\otimes(G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D})\mathrm{vec}\, J \\
+ \;& (M_{\Gamma^{-1}D}\otimes(G-\sigma^2 I_T)M_{\Gamma^{-1}D}LP_{\Gamma^{-1}D}\Gamma')\mathrm{vec}\, J' \\
+ \;& (M_{\Gamma^{-1}D}\Gamma'\otimes(G-\sigma^2 I_T)M_{\Gamma^{-1}D}L)\mathrm{vec}\, J,
\end{aligned}
$$

where the second equality is due to $(A\otimes B)(C\otimes D)=(AC\otimes BD)$. This expression can be further simplified. Indeed, use of $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\, B$ yields

$$
\begin{aligned}
&\mathrm{D}_\rho[(G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}] \\
&= -\mathrm{vec}\,(JS_y\Gamma^{-1\prime}M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}) - \mathrm{vec}\,(\Gamma^{-1}S_yJ'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}) \\
&+ \mathrm{vec}\,((G-\sigma^2 I_T)M_{\Gamma^{-1}D}J\Gamma P_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}) \\
&+ \mathrm{vec}\,((G-\sigma^2 I_T)P_{\Gamma^{-1}D}\Gamma'J'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}) \\
&+ \mathrm{vec}\,((G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}J\Gamma P_{\Gamma^{-1}D}) \\
&+ \mathrm{vec}\,((G-\sigma^2 I_T)M_{\Gamma^{-1}D}LP_{\Gamma^{-1}D}\Gamma'J'M_{\Gamma^{-1}D}) \\
&+ \mathrm{vec}\,((G-\sigma^2 I_T)M_{\Gamma^{-1}D}LJ\Gamma M_{\Gamma^{-1}D}) \\
&= \mathrm{vec}\,(-J\Gamma\Gamma^{-1}S_y\Gamma^{-1\prime}M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} - \Gamma^{-1}S_y\Gamma^{-1\prime}\Gamma'J'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}J\Gamma P_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} + (G-\sigma^2 I_T)P_{\Gamma^{-1}D}\Gamma'J'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}J\Gamma P_{\Gamma^{-1}D} + (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LP_{\Gamma^{-1}D}\Gamma'J'M_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LJ\Gamma M_{\Gamma^{-1}D}) \\
&= \mathrm{vec}\,(-LGM_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} - GL'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LP_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} + (G-\sigma^2 I_T)P_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}LP_{\Gamma^{-1}D} + (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LP_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LLM_{\Gamma^{-1}D}) \\
&= \mathrm{vec}\,(-LGM_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} - GL'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&- (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} - (G-\sigma^2 I_T)M_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&- (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} - (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LLM_{\Gamma^{-1}D} + (G-\sigma^2 I_T)L'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}L + (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LL'M_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LLM_{\Gamma^{-1}D}) \\
&= \mathrm{vec}\,(-LGM_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} - GL'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&- (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}(L'+L)M_{\Gamma^{-1}D} \\
&- (G-\sigma^2 I_T)M_{\Gamma^{-1}D}(L'+L)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}L + (G-\sigma^2 I_T)L'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G-\sigma^2 I_T)M_{\Gamma^{-1}D}LL'M_{\Gamma^{-1}D} + 2(G-\sigma^2 I_T)M_{\Gamma^{-1}D}LLM_{\Gamma^{-1}D}),
\end{aligned}
$$

where the third equality is due to $\Gamma^{-1}S_y\Gamma^{-1\prime} = G$ and $J\Gamma = L$, whereas the fourth equality is due to $P_{\Gamma^{-1}D} = I_T - M_{\Gamma^{-1}D}$. Hence, by making use of $\mathrm{tr}\,(A'B) = (\mathrm{vec}\,A)'\mathrm{vec}\,B$, we get

$$
\begin{aligned}
&\mathrm{D}_\rho R_1 \\
&= \mathrm{D}_\rho \mathrm{tr}\,[(G - \sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}] \\
&= (\mathrm{vec}\,I_T)'\mathrm{D}_\rho[(G - \sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}] \\
&= (\mathrm{vec}\,I_T)'\mathrm{vec}\,(-LGM_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} - GL'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&- (G - \sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}(L' + L)M_{\Gamma^{-1}D} \\
&- (G - \sigma^2 I_T)M_{\Gamma^{-1}D}(L' + L)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G - \sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}L + (G - \sigma^2 I_T)L'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G - \sigma^2 I_T)M_{\Gamma^{-1}D}LL'M_{\Gamma^{-1}D} + 2(G - \sigma^2 I_T)M_{\Gamma^{-1}D}LLM_{\Gamma^{-1}D}) \\
&= \mathrm{tr}\,(-GM_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}L - GL'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&- (G - \sigma^2 I_T)M_{\Gamma^{-1}D}(L' + L)M_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D} \\
&- (G - \sigma^2 I_T)M_{\Gamma^{-1}D}(L' + L)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G - \sigma^2 I_T)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}L + (G - \sigma^2 I_T)L'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&+ (G - \sigma^2 I_T)M_{\Gamma^{-1}D}LL'M_{\Gamma^{-1}D} + 2(G - \sigma^2 I_T)M_{\Gamma^{-1}D}LLM_{\Gamma^{-1}D}) \\
&= \mathrm{tr}\,(-\sigma^2 M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}L - \sigma^2 L'M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D} \\
&- (G - \sigma^2 I_T)M_{\Gamma^{-1}D}(L' + L)M_{\Gamma^{-1}D}(L' + L)M_{\Gamma^{-1}D} \\
&+ (G - \sigma^2 I_T)M_{\Gamma^{-1}D}LL'M_{\Gamma^{-1}D} + 2(G - \sigma^2 I_T)M_{\Gamma^{-1}D}LLM_{\Gamma^{-1}D}) \\
&= \mathrm{tr}\,(-\sigma^2 M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}(L' + L) + (G - \sigma^2 I_T)M_{\Gamma^{-1}D}LLM_{\Gamma^{-1}D} \\
&- (G - \sigma^2 I_T)M_{\Gamma^{-1}D}(L' + L)M_{\Gamma^{-1}D}(L' + L)M_{\Gamma^{-1}D} \\
&+ (G - \sigma^2 I_T)M_{\Gamma^{-1}D}L(L' + L)M_{\Gamma^{-1}D}) = r_1.
\end{aligned}
$$

Next up is $\mathrm{D}_\rho R_2$. We have

$$
\begin{aligned}
\mathrm{D}_\rho R_2 &= \mathrm{D}_\rho \mathrm{tr}\,[\sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L' + L)M_{\Gamma^{-1}D}G\Gamma^{-1}D] \\
&= (\mathrm{vec}\,I_T)'\mathrm{D}_\rho[\sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L' + L)M_{\Gamma^{-1}D}G\Gamma^{-1}D].
\end{aligned}
$$

From $\mathrm{d}_x\, AB = (\mathrm{d}_x\, A)B + A(\mathrm{d}_x\, B)$, we get

$$
\begin{aligned}
&\mathrm{d}_\rho[\sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L' + L)M_{\Gamma^{-1}D}G\Gamma^{-1}D] \\
&= \sigma^2\mathrm{d}_\rho[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}]D'\Gamma^{-1\prime}(L' + L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
&+ \sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}\mathrm{d}_\rho[D'\Gamma^{-1\prime}(L' + L)M_{\Gamma^{-1}D}G\Gamma^{-1}D]
\end{aligned}
$$

which via $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\, B$ yields

$$\begin{aligned}
&\mathrm{D}_\rho R_2 \\
&= \sigma^2(\mathrm{vec}\, I_T)'(D'\Gamma^{-1\prime} G M_{\Gamma^{-1}D}(L' + L)\Gamma^{-1}D \otimes I_m)\mathrm{D}_\rho[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}] \\
&+ \sigma^2(\mathrm{vec}\, I_T)'(I_m \otimes (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1})\mathrm{D}_\rho[D'\Gamma^{-1\prime}(L' + L)M_{\Gamma^{-1}D}G\Gamma^{-1}D]. \qquad \text{(D.33)}
\end{aligned}$$

Consider the first term on the right. From $\mathrm{d}_x\, A^{-1} = -A^{-1}(\mathrm{d}_x\, A)A^{-1}$, $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\, B$, and the symmetry of G, we get

$$\begin{aligned}
&\mathrm{D}_\rho\, (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1} \\
&= -((D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1} \otimes (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1})\mathrm{D}_\rho\, (D'\Gamma^{-1\prime}G\Gamma^{-1}D).
\end{aligned}$$

From $\mathrm{d}_x\, AB = (\mathrm{d}_x\, A)B + A(\mathrm{d}_x\, B)$,

$$\begin{aligned}
&\mathrm{d}_\rho\, (D'\Gamma^{-1\prime}G\Gamma^{-1}D) \\
&= D'(\mathrm{d}_\rho\, \Gamma^{-1\prime})G\Gamma^{-1}D + D'\Gamma^{-1\prime}(\mathrm{d}_\rho\, G)\Gamma^{-1}D + D'\Gamma^{-1\prime}G(\mathrm{d}_\rho\, \Gamma^{-1})D \\
&= D'(\mathrm{d}_\rho\, \Gamma^{-1\prime})G\Gamma^{-1}D + D'\Gamma^{-1\prime}(\mathrm{d}_\rho\, G)\Gamma^{-1}D + D'\Gamma^{-1\prime}G(\mathrm{d}_\rho\, \Gamma^{-1})D,
\end{aligned}$$

and so, via $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\, B$,

$$\begin{aligned}
&\mathrm{D}_\rho\, (D'\Gamma^{-1\prime}G\Gamma^{-1}D) \\
&= (D'\Gamma^{-1\prime}G \otimes D')\mathrm{D}_\rho\, \Gamma^{-1\prime} + (D'\Gamma^{-1\prime} \otimes D'\Gamma^{-1\prime})\mathrm{D}_\rho\, G + (D' \otimes D'\Gamma^{-1\prime}G)\mathrm{D}_\rho\, \Gamma^{-1} \\
&= -(D'\Gamma^{-1\prime}G \otimes D')\mathrm{vec}\, J' - (D' \otimes D'\Gamma^{-1\prime}G)\mathrm{vec}\, J \\
&- (D'\Gamma^{-1\prime} \otimes D'\Gamma^{-1\prime})(\Gamma^{-1} \otimes J)\mathrm{vec}\, S_y - (D'\Gamma^{-1\prime} \otimes D'\Gamma^{-1\prime})(J \otimes \Gamma^{-1})\mathrm{vec}\, S_y \\
&= -(D'\Gamma^{-1\prime}G \otimes D')\mathrm{vec}\, J' - (D' \otimes D'\Gamma^{-1\prime}G)\mathrm{vec}\, J \\
&- (D'\Gamma^{-1\prime}\Gamma^{-1} \otimes D'\Gamma^{-1\prime}J)\mathrm{vec}\, S_y - (D'\Gamma^{-1\prime}J \otimes D'\Gamma^{-1\prime}\Gamma^{-1})\mathrm{vec}\, S_y,
\end{aligned}$$

where the second equality is obtained by inserting the expressions for $\mathrm{D}_\rho\, \Gamma^{-1}$, $\mathrm{D}_\rho\, \Gamma^{-1\prime}$ and $\mathrm{D}_\rho\, G$, while the last equality is due to $(A \otimes B)(C \otimes D) = (AC \otimes BD)$. Further use of

$\text{vec}(ABC) = (C' \otimes A)\text{vec}\, B$ yields

$$\begin{aligned}
&\mathrm{D}_\rho\,(D'\Gamma^{-1\prime}G\Gamma^{-1}D)\\
&= -\text{vec}\,(D'J'G\Gamma^{-1}D) - \text{vec}\,(D'\Gamma^{-1\prime}GJD)\\
&- \text{vec}\,(D'\Gamma^{-1\prime}JS_y\Gamma^{-1\prime}\Gamma^{-1}D) - \text{vec}\,(D'\Gamma^{-1\prime}\Gamma^{-1}S_yJ'\Gamma^{-1}D)\\
&= -\text{vec}\,(D'\Gamma^{-1\prime}\Gamma'J'G\Gamma^{-1}D) - \text{vec}\,(D'\Gamma^{-1\prime}GJ\Gamma\Gamma^{-1}D)\\
&- \text{vec}\,(D'\Gamma^{-1\prime}J\Gamma\Gamma^{-1}S_y\Gamma^{-1\prime}\Gamma^{-1}D) - \text{vec}\,(D'\Gamma^{-1\prime}\Gamma^{-1}S_y\Gamma^{-1\prime}\Gamma'J'\Gamma^{-1}D)\\
&= \text{vec}\,(-D'\Gamma^{-1\prime}L'G\Gamma^{-1}D - D'\Gamma^{-1\prime}GL\Gamma^{-1}D - D'\Gamma^{-1\prime}LG\Gamma^{-1}D - D'\Gamma^{-1\prime}GL'\Gamma^{-1}D)\\
&= \text{vec}\,(-D'\Gamma^{-1\prime}(L'+L)G\Gamma^{-1}D - D'\Gamma^{-1\prime}G(L'+L)\Gamma^{-1}D).
\end{aligned}$$

This result, together with $\text{vec}(ABC) = (C' \otimes A)\text{vec}\, B$, $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ and $\text{tr}\,(A'B) = (\text{vec}\, A)'\text{vec}\, B$, imply that the first term on the right-hand side of (D.33) can be written as

$$\begin{aligned}
&\sigma^2(\text{vec}\, I_T)'(D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}(L'+L)\Gamma^{-1}D \otimes I_m)\mathrm{D}_\rho[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}]\\
&= -\sigma^2(\text{vec}\, I_T)'(D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}(L'+L)\Gamma^{-1}D \otimes I_m)((D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}\\
&\otimes\ (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1})\mathrm{D}_\rho(D'\Gamma^{-1\prime}G\Gamma^{-1}D)\\
&= -\sigma^2(\text{vec}\, I_T)'(D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}(L'+L)\Gamma^{-1}D(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}\\
&\otimes\ (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1})\mathrm{D}_\rho(D'\Gamma^{-1\prime}G\Gamma^{-1}D)\\
&= -\sigma^2(\text{vec}\, I_T)'(D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}(L'+L)\Gamma^{-1}D(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}\\
&\otimes\ (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1})\text{vec}\,(-D'\Gamma^{-1\prime}(L'+L)G\Gamma^{-1}D - D'\Gamma^{-1\prime}G(L'+L)\Gamma^{-1}D)\\
&= \sigma^2(\text{vec}\, I_T)'\text{vec}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(D'\Gamma^{-1\prime}(L'+L)G\Gamma^{-1}D\\
&+\ D'\Gamma^{-1\prime}G(L'+L)\Gamma^{-1}D)(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D]\\
&= \sigma^2\text{tr}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(D'\Gamma^{-1\prime}(L'+L)G\Gamma^{-1}D + D'\Gamma^{-1\prime}G(L'+L)\Gamma^{-1}D)\\
&\times\ (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D]. \qquad \text{(D.34)}
\end{aligned}$$

In order to evaluate the second term on the right of (D.33), we need $\mathrm{D}_\rho[D'\Gamma^{-1\prime}(L' +$

$L)M_{\Gamma^{-1}D}G\Gamma^{-1}D]$. From $\mathrm{d}_x\, AB = (\mathrm{d}_x\, A)B + A(\mathrm{d}_x\, B)$ and $\mathrm{d}_\rho\, M_{\Gamma^{-1}D} = -\mathrm{d}_\rho\, P_{\Gamma^{-1}D}$, we have

$$\begin{aligned}
&\mathrm{d}_\rho\,(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D)\\
&= D'(\mathrm{d}_\rho\,\Gamma^{-1\prime})(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D + D'\Gamma^{-1\prime}\mathrm{d}_\rho\,(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D\\
&+ D'\Gamma^{-1\prime}(L'+L)(\mathrm{d}_\rho\, M_{\Gamma^{-1}D})G\Gamma^{-1}D + D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}(\mathrm{d}_\rho\, G)\Gamma^{-1}D\\
&+ D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G(\mathrm{d}_\rho\,\Gamma^{-1})D\\
&= D'(\mathrm{d}_\rho\,\Gamma^{-1\prime})(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D + D'\Gamma^{-1\prime}\mathrm{d}_\rho\,(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D\\
&- D'\Gamma^{-1\prime}(L'+L)(\mathrm{d}_\rho\, P_{\Gamma^{-1}D})G\Gamma^{-1}D + D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}(\mathrm{d}_\rho\, G)\Gamma^{-1}D\\
&+ D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G(\mathrm{d}_\rho\,\Gamma^{-1})D.
\end{aligned}$$

By using this and $\mathrm{vec}(ABC) = (C'\otimes A)\mathrm{vec}\, B$, we can show that

$$\begin{aligned}
&\mathrm{D}_\rho\,(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D)\\
&= (D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}(L'+L)\otimes D')\mathrm{D}_\rho\,\Gamma^{-1\prime} + (D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}\otimes D'\Gamma^{-1\prime})\mathrm{D}_\rho\,(L'+L)\\
&- (D'\Gamma^{-1\prime}G\otimes D'\Gamma^{-1\prime}(L'+L))\mathrm{D}_\rho\, P_{\Gamma^{-1}D} + (D'\Gamma^{-1\prime}\otimes D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D})\mathrm{D}_\rho\, G\\
&+ (D'\otimes D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G)\mathrm{D}_\rho\,\Gamma^{-1},
\end{aligned}\tag{D.35}$$

where, via $\mathrm{D}_\rho\, L' = (L\otimes\Gamma')\mathrm{vec}\, J'$,

$$\mathrm{D}_\rho\,(L'+L) = \mathrm{D}_\rho\, L' + \mathrm{D}_\rho\, L = (L\otimes\Gamma')\mathrm{vec}\, J' + (\Gamma'\otimes L)\mathrm{vec}\, J.$$

By inserting this together with the expressions obtained for $\mathrm{D}_\rho\,\Gamma^{-1}$, $\mathrm{D}_\rho\,\Gamma^{-1\prime}$, $\mathrm{D}_\rho\, G$ and $\mathrm{D}_\rho\, P_{\Gamma^{-1}D}$ into (D.35), and using $(A\otimes B)(C\otimes D) = (AC\otimes BD)$, we obtain the following:

$$\begin{aligned}
&\mathrm{D}_\rho\,(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D)\\
&= -(D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}(L'+L)\otimes D')\mathrm{vec}J' - (D'\otimes D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G)\mathrm{vec}J\\
&+ (D'\Gamma^{-1\prime}GP_{\Gamma^{-1}D}\Gamma'\otimes D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D})\mathrm{vec}\, J\\
&+ (D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}\otimes D'\Gamma^{-1\prime}(L'+L)P_{\Gamma^{-1}D}\Gamma')\mathrm{vec}\, J'\\
&- (D'\Gamma^{-1\prime}\Gamma^{-1}\otimes D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}J)\mathrm{vec}\, S_y\\
&- (D'\Gamma^{-1\prime}J\otimes D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}\Gamma^{-1})\mathrm{vec}\, S_y\\
&+ (D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}L\otimes D'\Gamma^{-1\prime}\Gamma')\mathrm{vec}\, J'\\
&+ (D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}\Gamma'\otimes D'\Gamma^{-1\prime}L)\mathrm{vec}\, J,
\end{aligned}$$

which can be simplified using $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\, B$, $\Gamma^{-1}S_y\Gamma^{-1\prime} = G$ and $J\Gamma = L$;

$$
\begin{aligned}
& \mathrm{D}_\rho \left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
= \;& -\mathrm{vec}(D'\Gamma^{-1\prime}\Gamma'J'(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D) - \mathrm{vec}(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}GJ\Gamma\Gamma^{-1}D) \\
+ \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}J\Gamma P_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
+ \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)P_{\Gamma^{-1}D}\Gamma'J'M_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
- \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}J\Gamma\Gamma^{-1}S_y\Gamma^{-1\prime}\Gamma^{-1}D\right) \\
- \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}\Gamma^{-1}S_y\Gamma^{-1\prime}\Gamma'J'\Gamma^{-1}D\right) \\
+ \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}\Gamma'J'L'M_{\Gamma^{-1}D}G\Gamma^{-1}D\right) + \mathrm{vec}\left(D'\Gamma^{-1\prime}LJ\Gamma M_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
= \;& -\mathrm{vec}(D'\Gamma^{-1\prime}L'(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D) - \mathrm{vec}(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}GL\Gamma^{-1}D) \\
+ \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}LP_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
+ \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)P_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
- \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}LG\Gamma^{-1}D\right) - \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}GL'\Gamma^{-1}D\right) \\
+ \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}L'L'M_{\Gamma^{-1}D}G\Gamma^{-1}D\right) + \mathrm{vec}\left(D'\Gamma^{-1\prime}LLM_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
= \;& -\mathrm{vec}(D'\Gamma^{-1\prime}L'LM_{\Gamma^{-1}D}G\Gamma^{-1}D) + \mathrm{vec}\left(D'\Gamma^{-1\prime}LLM_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
- \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}LM_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
- \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}L'M_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
+ \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}LG\Gamma^{-1}D\right) + \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)L'M_{\Gamma^{-1}D}G\Gamma^{-1}D\right) \\
- \;& \mathrm{vec}\left(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}LG\Gamma^{-1}D\right) - \mathrm{vec}(D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G(L'+L)\Gamma^{-1}D) \\
= \;& \mathrm{vec}(D'\Gamma^{-1\prime}(L'+L)(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
- \;& D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
- \;& D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G(L'+L)\Gamma^{-1}D - 2D'\Gamma^{-1\prime}L'LM_{\Gamma^{-1}D}G\Gamma^{-1}D).
\end{aligned}
$$

By using this result, $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}\, B$ and $\mathrm{tr}\,(A'B) = (\mathrm{vec}\, A)'\mathrm{vec}\, B$, the second term in (D.33) becomes

$$
\begin{aligned}
&\sigma^2(\mathrm{vec}\, I_T)'(I_m \otimes (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1})\mathrm{D}_\rho[D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D] \\
= \;& \sigma^2(\mathrm{vec}\, I_T)'(I_m \otimes (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}) \\
\times \;& \mathrm{vec}(D'\Gamma^{-1\prime}(L'+L)(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
- \;& D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
- \;& D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G(L'+L)\Gamma^{-1}D - 2D'\Gamma^{-1\prime}L'LM_{\Gamma^{-1}D}G\Gamma^{-1}D) \\
= \;& \sigma^2(\mathrm{vec}\, I_T)'\mathrm{vec}[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(D'\Gamma^{-1\prime}(L'+L)(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
- \;& D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
- \;& D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G(L'+L)\Gamma^{-1}D - 2D'\Gamma^{-1\prime}L'LM_{\Gamma^{-1}D}G\Gamma^{-1}D)] \\
= \;& \sigma^2\mathrm{tr}[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(D'\Gamma^{-1\prime}(L'+L)(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
- \;& D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
- \;& D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G(L'+L)\Gamma^{-1}D - 2D'\Gamma^{-1\prime}L'LM_{\Gamma^{-1}D}G\Gamma^{-1}D)]. \qquad \text{(D.36)}
\end{aligned}
$$

Hence, by adding the results in (D.34) and (D.36), we get

$$
\begin{aligned}
&\mathrm{D}_\rho R_2 \\
= \;& \sigma^2(\mathrm{vec}\, I_T)'(D'\Gamma^{-1\prime}GM_{\Gamma^{-1}D}(L'+L)\Gamma^{-1}D \otimes I_m)\mathrm{D}_\rho[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}] \\
+ \;& \sigma^2(\mathrm{vec}\, I_T)'(I_m \otimes (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1})\mathrm{D}_\rho[D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D] \\
= \;& \sigma^2\mathrm{tr}\,[(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(D'\Gamma^{-1\prime}(L'+L)G\Gamma^{-1}D + D'\Gamma^{-1\prime}G(L'+L)\Gamma^{-1}D) \\
\times \;& (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
+ \;& (D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}(D'\Gamma^{-1\prime}(L'+L)(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
- \;& D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}(L'+L)M_{\Gamma^{-1}D}G\Gamma^{-1}D \\
- \;& D'\Gamma^{-1\prime}(L'+L)M_{\Gamma^{-1}D}G(L'+L)\Gamma^{-1}D - 2D'\Gamma^{-1\prime}L'LM_{\Gamma^{-1}D}G\Gamma^{-1}D)] = r_2, \qquad \text{(D.37)}
\end{aligned}
$$

which establishes (b). ■

E Proofs

In this section, for clarity, $\Gamma(1)$, $\Gamma(\rho_0)$, $L(1)$ and $L(\rho_0)$ will be denoted Γ_1, Γ_0, L_1 and L_0, respectively. Define $\phi_0(r) = \exp(r\alpha c_0)$ and $\rho = \exp(cN^{-\eta}T^{-\gamma}) = \exp(cT^{-1}\alpha)$ for some

$c \in \mathbb{C}$. Lemma E.1 is stated in terms of the following quantities:

$$\begin{aligned} h_0 &= [\phi_0(2) - 1 - 2\alpha c_0]/(4\alpha^2 c_0^2), \\ h_j(c) &= h_{j1} + (c_0 - c)\alpha h_{j2} + (c_0 - c)^2 \alpha^2 h_{j3} \end{aligned}$$

for $j \in \{1, ..., 5\}$ with

$$\begin{aligned}
h_{11} &= 1 - c_0\alpha + c_0^2\alpha^2/3, \\
h_{12} &= 1 - 2c_0\alpha/3, \\
h_{13} &= 1/3, \\
h_{21} &= 1/2 - c_0\alpha/3, \\
h_{22} &= [2c_0^3\alpha^3 + 6(\phi_0(1) - 1) - 3c_0\alpha(1 + \phi_0(1))]/(3c_0^3\alpha^3), \\
h_{23} &= [6 - 3c_0^2\alpha^2 - 2c_0^3\alpha^3 + 6(c_0\alpha - 1)\phi_0(1)]/(6c_0^4\alpha^4), \\
h_{31} &= [2c_0^3\alpha^3 - 12c_0\alpha\phi_0(1)(\phi_0(1) - 1) + 12(\phi_0(1) - 1)^2 + 3c_0^2\alpha^2(\phi_0(2) - 1)]/(6c_0^3\alpha^3), \\
h_{32} &= 2[-2c_0^3\alpha^3 - 3c_0^2\alpha^2\phi_0(2) - 6\phi_0(1)(\phi_0(1) - 1) + c_0\alpha(3 - 6\phi_0(1) + 9\phi_0(2))]/(6c_0^4\alpha^4), \\
h_{33} &= [-3 + 3c_0^2\alpha^2 + 2c_0^3\alpha^3 + 3(c_0\alpha - 1)^2\phi_0(2)]/(6c_0^5\alpha^5), \\
h_{41} &= [c_0^3\alpha^3 + 6(\phi_0(1) - 1) - 3c_0\alpha(1 + \phi_0(1))]/(3c_0^3\alpha^3), \\
h_{42} &= -[2\alpha^3c_0^3 + 24(\phi_0(1) - 1) + 3\alpha^2c_0^2(1 + \phi_0(1)) - 6\alpha c_0(1 + 3\phi_0(1))]/(3\alpha^4c_0^4), \\
h_{43} &= [-12 + 3c_0^2\alpha^2 + c_0^3\alpha^3 + 3(c_0\alpha - 2)^2\phi_0(1)]/(2c_0^5\alpha^5), \\
h_{51} &= 1/3, \\
h_{52} &= [6 - 3c_0^2\alpha^2 - 2c_0^3\alpha^3 + 6(c_0\alpha - 1)\phi_0(1)]/(3c_0^4\alpha^4), \\
h_{53} &= [-3 + 6\alpha c_0 + 6\alpha^2c_0^2 + 2\alpha^3c_0^3 - 12\alpha c_0\phi_0(1) + 3\phi_0(2)]/(6\alpha^5c_0^5),
\end{aligned}$$

which are all $O(1)$.

Lemma E.1. *Suppose that $D_t = (1,t)'$. Then, under Assumptions 2 and 3, uniformly in c,*

$$\text{(a)} \quad T^{-2}\text{tr}\,(L_0 L_0') = h_0 + O(T^{-1}),$$

$$\text{(b)} \quad N_T Q N_T = \left[\begin{array}{cc} 1 & 0 \\ 0 & h_1(c) \end{array}\right] + O(T^{-1/2}),$$

$$\text{(c)} \quad T^{-1} N_T D'\Gamma^{-1\prime} L_0 \Gamma^{-1} D N_T = \left[\begin{array}{cc} 0 & 0 \\ 0 & h_2(c) \end{array}\right] + O(T^{-1/2}),$$

$$\text{(d)} \quad T^{-2} N_T D'\Gamma^{-1\prime} L_0 L_0' \Gamma^{-1} D N_T = \left[\begin{array}{cc} 0 & 0 \\ 0 & h_3(c) \end{array}\right] + O(T^{-1/2}),$$

$$\text{(e)} \quad T^{-2} N_T D'\Gamma^{-1\prime} L_0 L_0 \Gamma^{-1} D N_T = \left[\begin{array}{cc} 0 & 0 \\ 0 & h_4(c) \end{array}\right] + O(T^{-1/2}),$$

$$\text{(f)} \quad T^{-2} N_T D'\Gamma^{-1\prime} L_0' L_0 \Gamma^{-1} D N_T = \left[\begin{array}{cc} 0 & 0 \\ 0 & h_5(c) \end{array}\right] + O(T^{-1/2}),$$

where $N_T = \text{diag}(1, T^{-1/2})$.

Proof: Consider (a). From the definition of α_q, $\rho_0 = \exp(c_0 N^{-\eta} T^{-\gamma}) = \exp(T^{-1}\alpha c_0)$. By using this result and the fact that $|tT^{-1} - r| = O(T^{-1})$, we can show that $|\exp(tT^{-1}) - \exp(r)| = O(T^{-1})$ uniformly in t and $r \in [0,1]$ (see, for example, Moon and Phillips, 2000, page 992). Hence, letting $t = \lfloor rT \rfloor$ and $\phi_0(r) = \exp(r\alpha c_0)$, we have

$$\rho_0^t = \exp(T^{-1} t\alpha c_0) = \phi_0(r) + O(T^{-1}), \tag{E.38}$$

and so

$$\begin{aligned} T^{-2}\text{tr}\,(L_0 L_0') &= \frac{1}{T^2}\sum_{t=0}^{T-2}(T-t-1)\rho_0^{2t} = \int_{r=0}^{1}(1-r)\phi_0(2r)dr + O(T^{-1}) \\ &= h_0 + O(T^{-1}), \end{aligned} \tag{E.39}$$

where by Taylor expansion of the type $\exp(x) = \sum_{j=0}^{\infty} x^j/j!$,

$$\begin{aligned} h_0 &= [\phi_0(2) - 1 - 2\alpha c_0]/(4\alpha^2 c_0^2) = \frac{1}{4\alpha^2 c_0^2}\left(\sum_{j=0}^{\infty}\frac{(2\alpha c_0)^j}{j!} - 1 - 2\alpha c_0\right) \\ &= \sum_{j=0}^{\infty}\frac{(2\alpha c_0)^j}{(j+2)!}. \end{aligned} \tag{E.40}$$

According to the ratio test, if $\lim_{j\to\infty}|a_{j+1}/a_j| < 1$, then $\sum_{j=0}^{\infty} a_j$ is convergent. Hence, since

$$\left|\frac{(\alpha c_0)^{j+1}/(j+1+k)!}{(\alpha c_0)^j/(j+k)!}\right| = \left|\frac{(j+k)!(\alpha c_0)^{j+1}}{(j+1+k)!(\alpha c_0)^j}\right| \le \left|\frac{1}{(j+1+k)}\right||\alpha c_0| \to 0$$

as $j \to \infty$ for $\alpha = O(1)$ and any finite k, $\sum_{j=0}^{\infty}(\alpha c_0)^j/(j+k)!$ converges. It follows that under these conditions, $h_0 = O(1)$. Regardless, we have

$$T^{-2}\text{tr}\,(L_0 L_0') = h_0 + O(T^{-1}), \tag{E.41}$$

which holds uniformly in c, because the remainder only depends on c_0. This establishes (a).

The result in (b) requires more work. We start by noting that $\Gamma^{-1} = \Gamma_0^{-1} + (\rho_0 - \rho)J$, leading to the following expression for Q:

$$\begin{aligned} Q &= D'\Gamma^{-1\prime}\Gamma^{-1}D \\ &= D'\Gamma_0^{-1\prime}\Gamma_0^{-1}D + (\rho_0-\rho)(D'\Gamma_0^{-1\prime}JD + D'J'\Gamma_0^{-1}D) + (\rho_0-\rho)^2 D'J'JD. \end{aligned} \tag{E.42}$$

Note how

$$\Gamma_0^{-1} = I_T - \rho_0 J = I_T - J - (\rho_0 - 1)J = \Gamma_1^{-1} - (\rho_0-1)J, \tag{E.43}$$

where $\Gamma_1^{-1} = \Gamma(1)^{-1} = I_T - J$. This means that

$$\begin{aligned} D'\Gamma_0^{-1\prime}\Gamma_0^{-1}D &= D'[\Gamma_1^{-1} - (\rho_0-1)J]'[\Gamma_1^{-1} - (\rho_0-1)J]D \\ &= D'\Gamma_1^{-1\prime}\Gamma_1^{-1}D - (\rho_0-1)(D'J'\Gamma_1^{-1}D + D'\Gamma_1^{-1\prime}JD) + (\rho_0-1)^2 D'J'JD. \end{aligned}$$

It is convenient to define $e_t = (0,...,0,1,0,...,0)'$, where the one sits at position t, and $E_t = 1_T - e_t = (1,...,1,0,1,...,1)'$. In this notation,

$$\begin{aligned} N_T D'\Gamma_1^{-1\prime}\Gamma_1^{-1}DN_T &= \begin{bmatrix} e_1'e_1 & T^{-1/2}e_1'1_T \\ T^{-1/2}1_T'e_1 & T^{-1}1_T'1_T \end{bmatrix} = \begin{bmatrix} 1 & T^{-1/2} \\ T^{-1/2} & 1 \end{bmatrix}, \\ N_T D'J'\Gamma_1^{-1}DN_T &= \begin{bmatrix} E_1'e_1 & T^{-1/2}E_1'1_T \\ T^{-1/2}t_T'J'e_1 & T^{-1}t_T'J'1_T \end{bmatrix} = \begin{bmatrix} 0 & T^{-1/2}(T-1) \\ 0 & T^{-1}\sum_{t=1}^{T-1} t \end{bmatrix} \\ &= \begin{bmatrix} 0 & T^{-1/2}(T-1) \\ 0 & (T-1)/2 \end{bmatrix}, \end{aligned}$$

and, since $J1_T = E_1$,

$$\begin{aligned} &N_T D'J'JDN_T \\ &= \begin{bmatrix} 1_T'J'J1_T & T^{-1/2}1_T'J'Jt_T \\ T^{-1/2}t_T'J'J1_T & T^{-1}t_T'J'Jt_T \end{bmatrix} = \begin{bmatrix} E_1'E_1 & T^{-1/2}E_1'Jt_T \\ T^{-1/2}t_T'J'E_1 & T^{-1}t_T'J'Jt_T \end{bmatrix} \\ &= \begin{bmatrix} T-1 & T^{-1/2}\sum_{t=1}^{T-1} t \\ T^{-1/2}\sum_{t=1}^{T-1} t & T^{-1}\sum_{t=1}^{T-1} t^2 \end{bmatrix} = \begin{bmatrix} T-1 & \sqrt{T}(T-1)/2 \\ \sqrt{T}(T-1)/2 & (T-1)[2(T-1)+1]/6 \end{bmatrix}. \end{aligned}$$

By using these results and the fact that $\exp(x) = 1 + x + O(x^2)$, we obtain

$$\begin{aligned} N_T D'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_T &= D'\Gamma_1^{-1\prime}\Gamma_1^{-1}D - (\rho_0-1)(D'J'\Gamma_1^{-1}D - D'\Gamma_1^{-1\prime}JD) + (\rho_0-1)^2 D'J'JD \\ &= N_T D'\Gamma_1^{-1\prime}\Gamma_1^{-1}DN_T - c_0\alpha T^{-1}N_T(D'J'\Gamma_1^{-1}D + D'\Gamma_1^{-1\prime}JD)N_T \\ &+ c_0^2\alpha^2 T^{-2}N_T D'J'JDN_T + O(\alpha^2 T^{-1}) + O(\alpha^3 T^{-1}). \end{aligned} \tag{E.44}$$

A word on the order of the remainder. Note how $T^{-1}\|N_T(D'J'\Gamma_1^{-1}D - D'\Gamma_1^{-1\prime}JD)N_T\| = O(1)$. The first of the two order terms is due to the error caused by the Taylor approximation of $(\rho_0 - 1)$, which is

$$O(\alpha^2 T^{-2})\|N_T(D'J'\Gamma_1^{-1}D - D'\Gamma_1^{-1\prime}JD)N_T\| = O(\alpha^2 T^{-1}).$$

The second order term is due to the approximation of $(\rho_0 - 1)^2$, which leads to an error whose order is of the form

$$O(\alpha^2(\alpha T^{-1} + \alpha^2 T^{-2} + ...)) = O(\alpha^2 T^{-1}(\alpha + \alpha^2 T^{-1} + \alpha^3 T^{-2} + ...)) = O(\alpha^3 T^{-1}),$$

where the last equality holds, because $\alpha T^{-1/2} = O(T^{-1/2}) = o(1)$. But then $O(\alpha^2 T^{-1}) + O(\alpha^3 T^{-1}) = O(T^{-1})$, and so

$$\begin{aligned} N_T D'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_T &= D'\Gamma_1^{-1\prime}\Gamma_1^{-1}D - (\rho_0 - 1)(D'J'\Gamma_1^{-1}D - D'\Gamma_1^{-1\prime}JD) + (\rho_0 - 1)^2 D'J'JD \\ &= N_T D'\Gamma_1^{-1\prime}\Gamma_1^{-1}DN_T - c_0\alpha T^{-1}N_T(D'J'\Gamma_1^{-1}D + D'\Gamma_1^{-1\prime}JD)N_T \\ &+ c_0^2\alpha^2 T^{-2}N_T D'J'JDN_T + O(T^{-1}). \end{aligned} \tag{E.45}$$

Many of the terms that we will be considering have the same form as $N_T D'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_T$. The evaluation of the remainder in these expressions will therefore be very similar to the one just given.

The above results for $N_T D'\Gamma_1^{-1\prime}\Gamma_1^{-1}DN_T$, $N_T D'J'\Gamma_1^{-1}DN_T$ and $N_T D'J'JDN_T$ imply

$$\begin{aligned} N_T D'\Gamma_1^{-1\prime}\Gamma_1^{-1}DN_T &= I_2 + O(T^{-1/2}), \\ T^{-1}N_T D'J'\Gamma_1^{-1}DN_T &= \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix} + O(T^{-1/2}), \\ T^{-2}N_T D'J'JDN_T &= \begin{bmatrix} 0 & 0 \\ 0 & 1/3 \end{bmatrix} + O(T^{-1/2}). \end{aligned}$$

Direct insertion in the expression for $N_T D'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_T$ yields

$$\begin{aligned} &N_T D'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_T \\ &= N_T D'\Gamma_1^{-1\prime}\Gamma_1^{-1}DN_T - c_0\alpha T^{-1}N_T(D'J'\Gamma_1^{-1}D + D'\Gamma_1^{-1\prime}JD)N_T \\ &+ c_0^2\alpha^2 T^{-2}N_T D'J'JDN_T + O(T^{-1}) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 - c_0\alpha + c_0^2\alpha^2/3 \end{bmatrix} + O(T^{-1/2}). \end{aligned} \tag{E.46}$$

Next, consider $N_T(D'\Gamma_0^{-1\prime}JD + D'J'\Gamma_0^{-1}D)N_T$. By using $\Gamma_0^{-1} = \Gamma_1^{-1} - (\rho_0 - 1)J$, Taylor expansion of ρ_0 around unity, and the above results for $N_T D'J'\Gamma_1^{-1}DN_T$ and $T^{-2}N_T D'J'JDN_T$,

$$\begin{aligned} T^{-1}N_T D'\Gamma_0^{-1\prime}JDN_T &= T^{-1}N_T D'\Gamma_1^{-1\prime}JDN_T - (\rho_0 - 1)T^{-1}N_T D'J'JDN_T \\ &= T^{-1}N_T D'\Gamma_1^{-1\prime}JDN_T - c_0\alpha T^{-2}N_T D'J'JDN_T + O(T^{-1}) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1/2 - c_0\alpha/3 \end{bmatrix} + O(T^{-1/2}). \end{aligned} \tag{E.47}$$

In view of $(\rho_0 - \rho) = (c_0 - c)\alpha T^{-1} + O(T^{-2})$, the above results imply

$$\begin{aligned} & N_T Q N_T \\ = \;& N_T D'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_T + (\rho_0 - \rho)N_T(D'\Gamma_0^{-1\prime}JD + D'J'\Gamma_0^{-1}D)N_T + (\rho_0 - \rho)^2 N_T D'J'JDN_T \\ = \;& N_T D'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_T + (c_0 - c)\alpha T^{-1}N_T(D'\Gamma_0^{-1\prime}JD + D'J'\Gamma_0^{-1}D)N_T \\ + \;& (c_0 - c)^2\alpha^2 T^{-2}N_T D'J'JDN_T + O(T^{-1}) \\ = \;& \overline{Q} + O(T^{-1/2}). \end{aligned} \tag{E.48}$$

The remainder here is not independent of c. However, the part of the remainder that drives its order is independent of c. Therefore, the result holds uniformly in c (see, for example, Moon and Phillips, 2004, Proof of Lemma 3).

Let us now consider (c). We have

$$\begin{aligned} D'\Gamma^{-1\prime}L_0\Gamma^{-1}D &= D'\Gamma_0^{-1\prime}L_0\Gamma_0^{-1}D + (\rho_0 - \rho)(D'\Gamma_0^{-1\prime}L_0JD + D'J'L_0\Gamma_0^{-1}D) \\ &+ (\rho_0 - \rho)^2 D'J'L_0JD. \end{aligned} \tag{E.49}$$

Direct calculations reveal that

$$\begin{aligned} & T^{-2}N_T D'\Gamma_1^{-1\prime}L_0JDN_T \\ &= \begin{bmatrix} T^{-2}e_1'L_0E_1 & T^{-5/2}e_1'L_0Jt_T \\ T^{-5/2}E_1'L_0E_1 & T^{-3}E_1'L_0Jt_T \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ T^{-5/2}\sum_{t=0}^{T-3}\sum_{s=0}^{t}\rho_0^s & T^{-3}\sum_{t=1}^{T-2}\sum_{s=0}^{t}(t-s)\rho_0^s \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & \int_{r=0}^{1}\int_{u=0}^{r}(r-u)\phi_0(u)dudr \end{bmatrix} + O(T^{-1/2}) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & -[2 + 2c_0\alpha + c_0^2\alpha^2 - 2\phi_0(1)]/(2c_0^3\alpha^3) \end{bmatrix} + O(T^{-1/2}), \end{aligned}$$

and

$$
\begin{aligned}
& T^{-3}N_T D'J'L_0 JDN_T \\
&= \left[\begin{array}{cc} T^{-3}E_1'L_0E_1 & T^{-7/2}E_1'L_0Jt_T \\ T^{-7/2}t_T'J'L_0E_1 & T^{-4}t_T'J'L_0Jt_T \end{array}\right] \\
&= \left[\begin{array}{cc} T^{-3}\sum_{t=0}^{T-3}\sum_{s=0}^{t}\rho_0^s & T^{-7/2}\sum_{t=1}^{T-2}\sum_{s=0}^{T-2-t} t\rho_0^s \\ T^{-7/2}\sum_{t=2}^{T-1}\sum_{s=0}^{t-2} t\rho_0^s & T^{-4}\sum_{t=1}^{T-2}\sum_{s=0}^{T-2-t} t(t+1+s)\rho_0^s \end{array}\right] \\
&= \left[\begin{array}{cc} 0 & 0 \\ 0 & \int_{r=0}^{1}\int_{u=0}^{1-r} r(r+u)\phi_0(u)dudr \end{array}\right] + O(T^{-1/2}) \\
&= \left[\begin{array}{cc} 0 & 0 \\ 0 & [6-3c_0^2\alpha^2-2c_0^3\alpha^3+6(c_0\alpha-1)\phi_0(1)]/(6c_0^4\alpha^4) \end{array}\right] + O(T^{-1/2}).
\end{aligned}
$$

Therefore, since $\Gamma_0^{-1} = \Gamma_1^{-1} - (\rho_0 - 1)J$ and $(\rho_0 - 1) = c_0\alpha T^{-1} + O(T^{-2})$,

$$
\begin{aligned}
& T^{-2}N_T D'J'L_0'\Gamma_0^{-1}DN_T \\
&= T^{-2}N_T D'J'L_0'\Gamma_1^{-1}DN_T - c_0\alpha T^{-3}N_T D'J'L_0'JDN_T + O(T^{-1}) \\
&= \left[\begin{array}{cc} 0 & 0 \\ 0 & [c_0^3\alpha^3+6(\phi_0(1)-1)-3c_0\alpha(1+\phi_0(1))]/(3c_0^3\alpha^3) \end{array}\right] + O(T^{-1/2}). \qquad \text{(E.50)}
\end{aligned}
$$

For $T^{-1}N_T D'\Gamma^{-1\prime}L_0\Gamma^{-1}DN_T$, we use the fact that $J\Gamma = L$, implying $L_0\Gamma_0^{-1} = J$. Hence, in view of (E.47),

$$
T^{-1}N_T D'\Gamma_0^{-1\prime}L_0\Gamma_0^{-1}DN_T = T^{-1}N_T D'\Gamma_0^{-1\prime}JDN_T,
$$

which is known from before. Insertion and simplification now yields

$$
\begin{aligned}
& T^{-1}N_T D'\Gamma^{-1\prime}L_0\Gamma^{-1}DN_T \\
&= T^{-1}N_T D'\Gamma_0^{-1\prime}L_0\Gamma_0^{-1}DN_T + (\rho_0-\rho)T^{-1}N_T(D'\Gamma_0^{-1\prime}L_0JD + D'J'L_0\Gamma_0^{-1}D)N_T \\
&+ (\rho_0-\rho)^2T^{-1}N_T D'J'L_0JDN_T \\
&= T^{-1}N_T D'\Gamma_0^{-1\prime}L_0\Gamma_0^{-1}DN_T + (c_0-c)\alpha T^{-2}N_T(D'\Gamma_0^{-1\prime}L_0JD + D'J'JD)N_T \\
&+ (c_0-c)^2\alpha^2T^{-3}N_T D'J'L_0JDN_T + O(T^{-1}) \\
&= \left[\begin{array}{cc} 0 & 0 \\ 0 & h_2(c) \end{array}\right] + O(T^{-1/2}), \qquad \text{(E.51)}
\end{aligned}
$$

which holds uniformly in c. Recall $\phi_0(1) = \sum_{j=0}^{\infty}(\alpha c_0)^j/j!$. Insertion and simplification yields

$$\begin{aligned} h_{22} &= \frac{1}{3c_0^3\alpha^3}[2c_0^3\alpha^3 + 6(\phi_0(1) - 1) - 3c_0\alpha(1 + \phi_0(1))] \\ &= \frac{1}{3c_0^3\alpha^3}\left(2c_0^3\alpha^3 + 6\sum_{j=0}^{\infty}\frac{(\alpha c_0)^{j+1}}{(j+1)!} - 3c_0\alpha\left(1 + \sum_{j=0}^{\infty}\frac{(\alpha c_0)^j}{j!}\right)\right) \\ &= \frac{1}{3c_0^3\alpha^3}\left(2c_0^3\alpha^3 + 6\sum_{j=0}^{\infty}\frac{(\alpha c_0)^{j+3}}{(j+3)!} - 3c_0\alpha\sum_{j=0}^{\infty}\frac{(\alpha c_0)^{j+2}}{(j+2)!}\right) \\ &= \frac{1}{3}\left(2 + 6\sum_{j=0}^{\infty}\frac{(\alpha c_0)^j}{(j+3)!} - 3\sum_{j=0}^{\infty}\frac{(\alpha c_0)^j}{(j+2)!}\right), \end{aligned}$$

and

$$\begin{aligned} h_{23} &= \frac{1}{6c_0^4\alpha^4}[6 - 3c_0^2\alpha^2 - 2c_0^3\alpha^3 + 6(c_0\alpha - 1)\phi_0(1)] \\ &= \frac{\alpha^2}{6c_0^4\alpha^4}\left(6 - 3c_0^2\alpha^2 - 2c_0^3\alpha^3 + 6(c_0\alpha - 1)\sum_{j=0}^{\infty}\frac{(\alpha c_0)^j}{j!}\right) \\ &= \frac{1}{6c_0^4\alpha^4}\left(6c_0\alpha\sum_{j=0}^{\infty}\frac{(\alpha c_0)^{j+3}}{(j+3)!} - 6\sum_{j=0}^{\infty}\frac{(\alpha c_0)^{j+4}}{(j+4)!}\right) \\ &= \sum_{j=0}^{\infty}\frac{(\alpha c_0)^j}{(j+3)!} - \sum_{j=0}^{\infty}\frac{(\alpha c_0)^j}{(j+4)!}, \end{aligned}$$

which are both $O(1)$. This can be verified by using the ratio test.

For (d), we use

$$\begin{aligned} & T^{-2}N_T D'\Gamma_0^{-1\prime}L_0L_0'\Gamma_0^{-1}DN_T \\ &= T^{-2}N_T D'\Gamma_1^{-1\prime}L_0L_0'\Gamma_1^{-1}DN_T - c_0\alpha T^{-3}N_T(D'\Gamma_1^{-1\prime}L_0L_0'JD + D'J'L_0L_0'\Gamma_1^{-1}D)N_T \\ &+ c_0^2\alpha^2T^{-4}N_T D'J'L_0L_0'JDN_T + O(T^{-1}). \end{aligned} \tag{E.52}$$

We start with the first term on the right, which we write as

$$\begin{aligned} & T^{-2}N_T D'\Gamma^{-1\prime}L_0L_0'\Gamma^{-1}DN_T \\ &= T^{-2}N_T D'\Gamma_0^{-1\prime}L_0L_0'\Gamma_0^{-1}DN_T + (\rho_0 - \rho)T^{-2}N_T(D'\Gamma_0^{-1\prime}L_0L_0'JD + D'J'L_0L_0'\Gamma_0^{-1}D)N_T \\ &+ (\rho_0 - \rho)^2T^{-2}N_T D'J'L_0L_0'JDN_T. \end{aligned} \tag{E.53}$$

Here

$$
\begin{aligned}
& T^{-4} N_T D' J' L_0 L_0' J D N_T \\
&= \begin{bmatrix} T^{-4} E_1' L_0 L_0' E_1 & T^{-9/2} E_1' L_0 L_0' J t_T \\ T^{-9/2} t_T' J' L_0 L_0' E_1 & T^{-5} t_T' J' L_0 L_0' J t_T \end{bmatrix} \\
&= \begin{bmatrix} T^{-4} \sum_{t=0}^{T-2} (\sum_{s=0}^{t} \rho_0^s)^2 & T^{-9/2} \sum_{t=1}^{T-1} \sum_{k=0}^{T-t-1} \sum_{s=0}^{T-t-1} (t+k) \rho_0^{s+k} \\ T^{-9/2} \sum_{t=1}^{T-1} \sum_{k=0}^{T-t-1} \sum_{s=0}^{T-t-1} (t+k) \rho_0^{s+k} & T^{-5} \sum_{t=1}^{T-1} [\sum_{s=0}^{T-t-1} (t+s) \rho_0^s]^2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \int_{r=0}^{1} [\int_{u=0}^{1-r} (r+u) \phi_0(u) du]^2 dr \end{bmatrix} + O(T^{-1/2}) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & h_{33} \end{bmatrix} + O(T^{-1/2}),
\end{aligned}
$$

where

$$
\begin{aligned}
h_{33} &= \frac{1}{6 c_0^5 \alpha^5} [-3 + 3 c_0^2 \alpha^2 + 2 c_0^3 \alpha^3 + 3 (c_0 \alpha - 1)^2 \phi_0(2)] \\
&= \frac{1}{6 c_0^5 \alpha^5} \left(-3 + 3 c_0^2 \alpha^2 + 2 c_0^3 \alpha^3 + 3 (c_0 \alpha - 1)^2 \sum_{j=0}^{\infty} \frac{(2 \alpha c_0)^j}{j!} \right) \\
&= \frac{1}{6 c_0^5 \alpha^5} \left(3 c_0^2 \alpha^2 \sum_{j=0}^{\infty} \frac{(2 \alpha c_0)^{j+3}}{(j+3)!} - 6 c_0 \alpha \sum_{j=0}^{\infty} \frac{(2 \alpha c_0)^{j+4}}{(j+4)!} + 3 \sum_{j=0}^{\infty} \frac{(2 \alpha c_0)^{j+5}}{(j+5)!} \right) \\
&= \frac{1}{6} \left(3 \sum_{j=0}^{\infty} \frac{(2 \alpha c_0)^j}{(j+3)!} - 6 \sum_{j=0}^{\infty} \frac{(2 \alpha c_0)^j}{(j+4)!} + 3 \sum_{j=0}^{\infty} \frac{(2 \alpha c_0)^j}{(j+5)!} \right),
\end{aligned}
$$

which is $O(1)$ under $\alpha = O(1)$. By using this,

$$
\begin{aligned}
& T^{-2} N_T D' \Gamma_1^{-1\prime} L_0 L_0' \Gamma_1^{-1} D N_T \\
&= \begin{bmatrix} T^{-2} e_1' L_0 L_0' e_1 & T^{-5/2} e_1' L_0 L_0' E_1 \\ T^{-5/2} E_1' L_0 L_0' e_1 & T^{-3} E_1' L_0 L_0' E_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & T^{-3} \sum_{t=0}^{T-2} (\sum_{s=0}^{t} \rho_0^s)^2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \int_{r=0}^{1} [\int_{u=0}^{r} \phi_0(u) du]^2 dr \end{bmatrix} + O(T^{-1}) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & [3 + 2 c_0 \alpha - 4 \phi_0(1) + \phi_0(2)] / (2 c_0^3 \alpha^3) \end{bmatrix} + O(T^{-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
& T^{-3}N_T D'\Gamma_1^{-1\prime} L_0 L_0' J D N_T \\
&= \begin{bmatrix} T^{-3}e_1' L_0 L_0' E_1 & T^{-7/2}e_1' L_0 L_0' J t_T \\ T^{-7/2}E_1' L_0 L_0' E_1 & T^{-4}E_1' L_0 L_0' J t_T \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ T^{-7/2}\sum_{t=0}^{T-2}(\sum_{s=0}^{t}\rho_0^s)^2 & T^{-4}\sum_{t=1}^{T-1}\sum_{k=0}^{T-t-1}\sum_{s=0}^{T-t-1}(t+k)\rho_0^{s+k} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \int_{r=0}^{1}\int_{u=0}^{1-r}\int_{v=0}^{1-r}(r+u)\phi_0(u+v)dvdudr \end{bmatrix} + O(T^{-1/2}) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & [c_0^2\alpha^2 - (\phi_0(1)-1)^2 + c_0\alpha(\phi_0(1)-1)^2]/(2c_0^4\alpha^4) \end{bmatrix} + O(T^{-1/2}),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
& T^{-2}N_T D'\Gamma_0^{-1\prime} L_0 L_0' \Gamma_0^{-1} D N_T \\
&= T^{-2}N_T D'\Gamma_1^{-1\prime} L_0 L_0' \Gamma_1^{-1} D N_T - c_0\alpha T^{-3} N_T (D'\Gamma_1^{-1\prime} L_0 L_0' J D + D' J' L_0 L_0' \Gamma_1^{-1} D) N_T \\
&+ c_0^2\alpha^2 T^{-4} N_T D' J' L_0 L_0' J D N_T + O(\alpha_1^2 T^{-1}) + O(\alpha_1^3 T^{-1}) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & h_{31} \end{bmatrix} + O(T^{-1/2}), \qquad \text{(E.54)}
\end{aligned}
$$

where

$$
\begin{aligned}
h_{31} &= [3 + 2c_0\alpha - 4\phi_0(1) + \phi_0(2)]/(2c_0^3\alpha^3) \\
&- 2c_0\alpha[c_0^2\alpha^2 - (\phi_0(1)-1)^2 + c_0\alpha(\phi_0(1)-1)^2]/(2c_0^4\alpha^4) \\
&+ c_0^2\alpha^2[-3 + 3c_0^2\alpha^2 + 2c_0^3\alpha^3 + 3(c_0\alpha-1)^2\phi_0(2)]/(6c_0^5\alpha^5) \\
&= [2c_0^3\alpha^3 - 12c_0\alpha\phi_0(1)(\phi_0(1)-1) + 12(\phi_0(1)-1)^2 + 3c_0^2\alpha^2(\phi_0(2)-1)]/(6c_0^3\alpha^3).
\end{aligned}
$$

The usual approach can be used to show that $h_{31} = O(1)$ under $\alpha = O(1)$.

The same results can be used to show that

$$
\begin{aligned}
& T^{-3}N_T D'\Gamma_0^{-1\prime} L_0 L_0' J D \\
&= T^{-3}N_T D'\Gamma_1^{-1\prime} L_0 L_0' J D N_T - (\rho_0 - 1)T^{-3} N_T D' J' L_0 L_0' J D N_T \\
&= T^{-3}N_T D'\Gamma_1^{-1\prime} L_0 L_0' J D N_T - c_0\alpha T^{-4} N_T D' J' L_0 L_0' J D N_T + O(\alpha_1^2 T^{-1}) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & h_{32} \end{bmatrix} + O(T^{-1/2}), \qquad \text{(E.55)}
\end{aligned}
$$

where

$$
\begin{aligned}
h_{32} &= [c_0^2\alpha^2 - (\phi_0(1)-1)^2 + c_0\alpha(\phi_0(1)-1)^2]/(2c_0^4\alpha^4) \\
&- c_0\alpha[-3 + 3c_0^2\alpha^2 + 2c_0^3\alpha^3 + 3(c_0\alpha-1)^2\phi_0(2)]/(6c_0^5\alpha^5) \\
&= [-2c_0^3\alpha^3 - 3c_0^2\alpha^2\phi_0(2) - 6\phi_0(1)(\phi_0(1)-1) + c_0\alpha(3 - 6\phi_0(1) + 9\phi_0(2))]/(6c_0^4\alpha^4),
\end{aligned}
$$

which is again $O(1)$ under $\alpha = O(1)$. Hence, by putting everything together,

$$\begin{aligned}
& T^{-2}N_T D'\Gamma^{-1\prime}L_0L_0'\Gamma^{-1}DN_T \\
= \;& T^{-2}N_T D'\Gamma_0^{-1\prime}L_0L_0'\Gamma_0^{-1}DN_T + (\rho_0-\rho)T^{-2}N_T(D'\Gamma_0^{-1\prime}L_0L_0'JD + D'J'L_0L_0'\Gamma_0^{-1}D)N_T \\
+ \;& (\rho_0-\rho)^2T^{-2}N_TD'J'L_0L_0'JDN_T \\
= \;& T^{-2}N_T D'\Gamma_0^{-1\prime}L_0L_0'\Gamma_0^{-1}DN_T + (c_0-c)\alpha T^{-3}N_T(D'\Gamma_0^{-1\prime}L_0L_0'JD + D'J'L_0L_0'\Gamma_0^{-1}D)N_T \\
+ \;& (c_0-c)^2\alpha^2T^{-4}N_TD'J'L_0L_0'JDN_T + O(T^{-1}) \\
= \;& \begin{bmatrix} 0 & 0 \\ 0 & h_3(c) \end{bmatrix} + O(T^{-1/2}),
\end{aligned} \tag{E.56}$$

uniformly in c.

Next up is (e). We have

$$\begin{aligned}
& T^{-2}N_T D'\Gamma^{-1\prime}L_0L_0\Gamma^{-1}DN_T \\
= \;& T^{-2}N_T D'\Gamma_0^{-1\prime}L_0L_0\Gamma_0^{-1}DN_T + (\rho_0-\rho)T^{-2}N_T(D'\Gamma_0^{-1\prime}L_0L_0JD + D'J'L_0L_0\Gamma_0^{-1}D)N_T \\
+ \;& (\rho_0-\rho)^2T^{-2}N_TD'J'L_0L_0JDN_T \\
= \;& T^{-2}N_T D'\Gamma_0^{-1\prime}L_0JDN_T + (\rho_0-\rho)T^{-2}N_T(D'\Gamma_0^{-1\prime}L_0L_0JD + D'J'L_0JD)N_T \\
+ \;& (\rho_0-\rho)^2T^{-2}N_TD'J'L_0L_0JDN_T.
\end{aligned} \tag{E.57}$$

Here

$$\begin{aligned}
& T^{-4}N_TD'J'L_0L_0JDN_T \\
= \;& \begin{bmatrix} T^{-4}E_1'L_0L_0E_1 & T^{-9/2}E_1'L_0L_0Jt_T \\ T^{-9/2}t_T'J'L_0L_0E_1 & T^{-5}t_T'J'L_0L_0Jt_T \end{bmatrix} \\
= \;& \begin{bmatrix} T^{-4}\sum_{t=0}^{T-3}\sum_{s=0}^{t}\sum_{k=0}^{T-4-t}\rho_0^{s+k} & T^{-9/2}\sum_{t=1}^{T-3}\sum_{s=0}^{t}\sum_{k=0}^{T-t-4}(t-k+1)\rho_0^{s+k} \\ T^{-9/2}\sum_{t=2}^{T-2}\sum_{s=t}^{T-2}\sum_{k=0}^{t-2}(s+1)\rho_0^{s+k-t} & T^{-5}\sum_{t=2}^{T-2}\sum_{s=t}^{T-2}\sum_{k=1}^{t-1}(s+1)(t-k)\rho_0^{s+k-t-1} \end{bmatrix} \\
= \;& \begin{bmatrix} 0 & 0 \\ 0 & \int_{r=0}^{1}\int_{u=r}^{1}\int_{v=0}^{r}u(r-v)\phi_0(u+v-r)dvdudr \end{bmatrix} + O(T^{-1/2}) \\
= \;& \begin{bmatrix} 0 & 0 \\ 0 & h_{43} \end{bmatrix} + O(T^{-1/2}).
\end{aligned}$$

The limit of $T^{-2}N_TD'\Gamma_0^{-1\prime}L_0JDN_T$ is equal to the limit of $T^{-2}N_TD'J'L_0'\Gamma_0^{-1}DN_T$, as the latter has already been shown to be asymptotically symmetric. Hence, since we have already evaluated $T^{-3}N_T D' J' L_0 J D N_T$, it only remains to consider $T^{-3}N_T D'\Gamma_0^{-1\prime} L_0 L_0 J D N_T$. We have

$$\begin{aligned}
&T^{-3}N_T D'\Gamma_1^{-1\prime} L_0 L_0 J D N_T \\
&= \begin{bmatrix} T^{-3}e_1' L_0 L_0 E_1 & T^{-7/2}e_1' L_0 L_0 J t_T \\ T^{-7/2}E_1' L_0 L_0 E_1 & T^{-4}E_1' L_0 L_0 J t_T \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ T^{-7/2}\sum_{t=0}^{T-3}\sum_{s=0}^{t}\sum_{k=0}^{T-4-t}\rho_0^{s+k} & T^{-4}\sum_{t=1}^{T-3}\sum_{s=0}^{t}\sum_{k=0}^{T-t-4}(t-k+1)\rho_0^{s+k} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \int_{r=0}^{1}\int_{u=0}^{r}\int_{v=0}^{1-r}(r-v)\phi_0(u+v)dv du dr \end{bmatrix} + O(T^{-1/2}) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & [6-6\phi_0(1)+c_0\alpha(4+c_0\alpha+2\phi_0(1))]/(2c_0^4\alpha^4) \end{bmatrix} + O(T^{-1/2}).
\end{aligned}$$

Hence, since

$$\begin{aligned}
&[6-6\phi_0(1)+c_0\alpha(4+c_0\alpha+2\phi_0(1))]/(2c_0^4\alpha^4) - c_0\alpha h_{43} \\
&= -[2\alpha^3c_0^3+42(\phi_0(1)-1)+3\alpha^2c_0^2(1+2\phi_0(1))-6\alpha c_0(2+5\phi_0(1))]/(6\alpha^4c_0^4),
\end{aligned}$$

we obtain

$$\begin{aligned}
&T^{-3}N_T D'\Gamma_0^{-1\prime} L_0 L_0 J D \\
&= T^{-3}N_T D'\Gamma_1^{-1\prime} L_0 L_0 J D N_T - (\rho_0-1)T^{-3}N_T D' J' L_0 L_0 J D N_T \\
&= T^{-3}N_T D'\Gamma_1^{-1\prime} L_0 L_0 J D N_T - c_0\alpha T^{-4}N_T D' J' L_0 L_0 J D N_T + O(T^{-1}) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & -[2\alpha^3c_0^3+42(\phi_0(1)-1)+3\alpha^2c_0^2(1+2\phi_0(1))-6\alpha c_0(2+5\phi_0(1))]/(6\alpha^4c_0^4) \end{bmatrix} \\
&+ O(T^{-1/2}). \qquad \text{(E.58)}
\end{aligned}$$

This implies

$$\begin{aligned}
&T^{-2}N_T D'\Gamma^{-1\prime} L_0 L_0 \Gamma^{-1} D N_T \\
&= T^{-2}N_T D'\Gamma_0^{-1\prime} L_0 J D N_T + (\rho_0-\rho)T^{-2}N_T(D'\Gamma_0^{-1\prime} L_0 L_0 J D + D' J' L_0 J D)N_T \\
&+ (\rho_0-\rho)^2 T^{-2}N_T D' J' L_0 L_0 J D N_T \\
&= T^{-2}N_T D'\Gamma_0^{-1\prime} L_0 J D N_T + (c_0-c)\alpha T^{-3}N_T(D'\Gamma_0^{-1\prime} L_0 L_0 J D + D' J' L_0 J D)N_T \\
&+ (c_0-c)^2\alpha^2 T^{-4}N_T D' J' L_0 L_0 J D N_T + O(T^{-1}) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & h_4(c) \end{bmatrix} + O(T^{-1/2}), \qquad \text{(E.59)}
\end{aligned}$$

uniformly in c, where h_{42} in $h_4(c)$ is derived from

$$\begin{aligned}
h_{42} &= -[2\alpha^3c_0^3+42(\phi_0(1)-1)+3\alpha^2c_0^2(1+2\phi_0(1))-6\alpha c_0(2+5\phi_0(1))]/(6\alpha^4c_0^4) \\
&+ [6-3c_0^2\alpha^2-2c_0^3\alpha^3+6(c_0\alpha-1)\phi_0(1)]/(6c_0^4\alpha^4) \\
&= -[2\alpha^3c_0^3+24(\phi_0(1)-1)+3\alpha^2c_0^2(1+\phi_0(1))-6\alpha c_0(1+3\phi_0(1))]/(3\alpha^4c_0^4).
\end{aligned}$$

It remains to consider (f). In the usual fashion, we have

$$
\begin{aligned}
& T^{-2}N_T D'\Gamma^{-1\prime}L_0'L_0\Gamma^{-1}DN_T \\
= \;& T^{-2}N_T D'\Gamma_0^{-1\prime}L_0'L_0\Gamma_0^{-1}DN_T + (\rho_0-\rho)T^{-2}N_T(D'\Gamma_0^{-1\prime}L_0'L_0JD + D'J'L_0'L_0\Gamma_0^{-1}D)N_T \\
+ \;& (\rho_0-\rho)^2T^{-2}N_TD'J'L_0'L_0JDN_T \\
= \;& T^{-2}N_TD'J'JDN_T + (\rho_0-\rho)T^{-2}N_T(D'J'L_0JD + D'J'L_0'JD)N_T \\
+ \;& (\rho_0-\rho)^2T^{-2}N_TD'JL_0'L_0JDN_T \\
= \;& T^{-2}N_TD'J'JDN_T + (c_0-c)\alpha T^{-3}N_T(D'J'L_0JD + D'J'L_0'JD)N_T \\
+ \;& (c_0-c)^2\alpha^2T^{-4}N_TD'J'L_0'L_0JDN_T + O(T^{-1}), \qquad \text{(E.60)}
\end{aligned}
$$

uniformly in c. All terms here are known, except for $T^{-4}N_TD'J'L_0'L_0JDN_T$. A direct calculation reveals that

$$
\begin{aligned}
& T^{-4}N_TD'J'L_0'L_0JDN_T \\
= \;& \begin{bmatrix} T^{-4}E_1'L_0'L_0E_1 & T^{-9/2}E_1'L_0'L_0Jt_T \\ T^{-9/2}t_T'J'L_0'L_0E_1 & T^{-5}t_T'J'L_0'L_0Jt_T \end{bmatrix} \\
= \;& \begin{bmatrix} T^{-4}\sum_{t=0}^{T-3}(\sum_{s=0}^{t}\rho_0^s)^2 & T^{-9/2}\sum_{t=1}^{T-3}\sum_{s=0}^{t}\sum_{k=0}^{t}(t-k+1)\rho_0^{s+k} \\ T^{-9/2}\sum_{t=1}^{T-3}\sum_{s=0}^{t}\sum_{k=0}^{t}(t-k+1)\rho_0^{s+k} & T^{-5}\sum_{t=1}^{T-3}(\sum_{s=0}^{t}(t-s+1)\rho_0^s)^2 \end{bmatrix} \\
= \;& \begin{bmatrix} 0 & 0 \\ 0 & \int_{r=0}^{1}(\int_{u=0}^{r}(r-u)\phi_0(u)du)^2dr \end{bmatrix} + O(T^{-1/2}) \\
= \;& \begin{bmatrix} 0 & 0 \\ 0 & h_{53} \end{bmatrix} + O(T^{-1/2}),
\end{aligned}
$$

giving

$$
T^{-2}N_TD'\Gamma^{-1\prime}L_0'L_0\Gamma^{-1}DN_T = \begin{bmatrix} 0 & 0 \\ 0 & h_5(c) \end{bmatrix} + O(T^{-1/2}). \qquad \text{(E.61)}
$$

This establishes (f), and hence the proof of Lemma E.1 is complete. ■.

Proof of Lemma 1.

From the first-order condition with respect to S_λ we obtain the following slightly modified expression for $\hat{\Lambda}$:

$$
\hat{\Lambda} = I_T + \sigma^{-2}\Gamma^{-1}D\hat{S}_\lambda D'\Gamma^{-1\prime}.
$$

The Woodbury identity states that $(A+CBC')^{-1} = A^{-1} - A^{-1}C(B^{-1}+C'A^{-1}C)^{-1}C'A^{-1}$ (see Abadir and Magnus, 2005, Exercise 5.17). Application of this identity to $\hat{\Lambda}^{-1}$ yields, with $K = (\sigma^2 \hat{S}_\lambda^{-1} + Q)^{-1}$,

$$\hat{\Lambda}^{-1} = I_T - \Gamma^{-1} D(\sigma^2 \hat{S}_\lambda^{-1} + Q)^{-1} D'\Gamma^{-1\prime} = I_T - \Gamma^{-1} DKD'\Gamma^{-1\prime},$$

and therefore

$$Q^* = T\log(\sigma^2) + \log(|\hat{\Lambda}|) + \sigma^{-2}\mathrm{tr}\, G - \sigma^{-2}\mathrm{tr}\,(G\Gamma^{-1}DKD'\Gamma^{-1\prime}),$$

where $G = \Gamma^{-1} S_y \Gamma^{-1\prime}$ is as before.

Consider $\sigma^{-2}\mathrm{tr}\,(G\Gamma^{-1}DKD'\Gamma^{-1\prime})$. Clearly,

$$\begin{aligned} \hat{S}_\lambda &= \sigma^2(\Gamma^{-1}D)^{-}(\sigma^{-2}G - I_T)(\Gamma^{-1}D)^{-\prime} \\ &= (D'\Gamma^{-1\prime}\Gamma^{-1}D)^{-1} D'\Gamma^{-1\prime} G\Gamma^{-1} D(D'\Gamma^{-1\prime}\Gamma^{-1}D)^{-1} - \sigma^2(D'\Gamma^{-1\prime}\Gamma^{-1}D)^{-1} \\ &= Q^{-1}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1} - \sigma^2 Q^{-1}. \end{aligned} \quad \text{(E.62)}$$

By using this and $(A + CBC')^{-1} = A^{-1} - A^{-1}C(B^{-1} + C'A^{-1}C)^{-1}C'A^{-1}$,

$$\begin{aligned} K &= (\sigma^2 \hat{S}_\lambda^{-1} + Q)^{-1} = Q^{-1} - Q^{-1}(\sigma^{-2}\hat{S}_\lambda + Q^{-1})^{-1}Q^{-1} \\ &= Q^{-1} - \sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1}. \end{aligned} \quad \text{(E.63)}$$

Direct insertion now yields

$$\begin{aligned} &\mathrm{tr}\,(G\Gamma^{-1}DKD'\Gamma^{-1\prime}) \\ &= \mathrm{tr}\,(D'\Gamma^{-1\prime}G\Gamma^{-1}DK) = \mathrm{tr}\,[D'\Gamma^{-1\prime}G\Gamma^{-1}D(Q^{-1} - \sigma^2(D'\Gamma^{-1\prime}G\Gamma^{-1}D)^{-1})] \\ &= \mathrm{tr}\,(D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}) - \sigma^2\mathrm{tr}\, I_m = \mathrm{tr}\,(D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}) - \sigma^2 m, \end{aligned} \quad \text{(E.64)}$$

where m is the dimension of D_t.

Consider

$$D'\Gamma^{-1\prime}G\Gamma^{-1}D = D'\Gamma^{-1\prime}\Gamma^{-1}S_y\Gamma^{-1\prime}\Gamma^{-1}D. \quad \text{(E.65)}$$

Letting $S_\varepsilon = N^{-1}\sum_{i=1}^N \varepsilon_i\varepsilon_i'$, S_y can be expanded as follows:

$$\begin{aligned} S_y &= \Gamma_0 S_u \Gamma_0' = \Gamma_0 \frac{1}{N}\sum_{i=1}^N (\Gamma_0^{-1}D\lambda_i + \varepsilon_i)(\Gamma_0^{-1}D\lambda_i + \varepsilon_i)'\Gamma_0' \\ &= \sigma_0^2\Gamma_0\Gamma_0' + DS_\lambda D' + \frac{1}{N}\sum_{i=1}^N D\lambda_i\varepsilon_i'\Gamma_0' + \Gamma_0\frac{1}{N}\sum_{i=1}^N \varepsilon_i\lambda_i'D' + \Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'. \end{aligned} \quad \text{(E.66)}$$

This implies

$$
\begin{aligned}
& D'\Gamma^{-1\prime}G\Gamma^{-1}D \\
= \;& D'\Gamma^{-1\prime}\Gamma^{-1}S_y\Gamma^{-1\prime}\Gamma^{-1}D \\
= \;& \sigma_0^2 D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D + D'\Gamma^{-1\prime}\Gamma^{-1}DS_\lambda D'\Gamma^{-1\prime}\Gamma^{-1}D \\
+ \;& D'\Gamma^{-1\prime}\Gamma^{-1}\frac{1}{N}\sum_{i=1}^{N} D\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D + D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i' D'\Gamma^{-1\prime}\Gamma^{-1}D \\
+ \;& D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D \\
= \;& \sigma_0^2 D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D + QS_\lambda Q + Q\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D \\
+ \;& D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i' Q + D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D. \qquad \text{(E.67)}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\operatorname{tr}(G\Gamma^{-1}DKD'\Gamma^{-1\prime}) &= \operatorname{tr}(D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1}) - \sigma^2 m \\
&= \sigma_0^2\operatorname{tr}(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}) + \operatorname{tr}(QS_\lambda) \\
&+ 2\operatorname{tr}\left(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i'\right) \\
&+ \operatorname{tr}[D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}] - \sigma^2 m. \qquad \text{(E.68)}
\end{aligned}
$$

The same expansion can be used to show that

$$
\begin{aligned}
\operatorname{tr} G &= \operatorname{tr}(\Gamma^{-1}S_y\Gamma^{-1\prime}) \\
&= \sigma_0^2\operatorname{tr}(\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}) + \operatorname{tr}(QS_\lambda) + 2\operatorname{tr}\left(\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i' D'\Gamma^{-1\prime}\right) \\
&+ \operatorname{tr}[\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}]. \qquad \text{(E.69)}
\end{aligned}
$$

These expressions can be substituted into Q^*, giving, after cancellation of common terms,

$$
\begin{aligned}
Q^* &= T\log(\sigma^2) + \log(|\hat{\Lambda}|) + \sigma^{-2}\operatorname{tr} G - \sigma^{-2}\operatorname{tr}(G\Gamma^{-1}DKD'\Gamma^{-1\prime}) \\
&= T\log(\sigma^2) + \log(|\hat{\Lambda}|) + \sigma^{-2}\sigma_0^2\operatorname{tr}(\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}) \\
&- \sigma^{-2}\sigma_0^2\operatorname{tr}(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}) \\
&+ \sigma^{-2}\operatorname{tr}[\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}] \\
&- \sigma^{-2}\operatorname{tr}[D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}] + m, \qquad \text{(E.70)}
\end{aligned}
$$

where of course $D_t = (1, t)'$ in the current constant and trend case.

Consider $\operatorname{tr}(\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime})$. Note how $\Gamma^{-1}\Gamma_0 = I_T + (\rho_0 - \rho)L_0$. Since $\operatorname{tr} L_0 = 0$, this implies

$$\begin{aligned}\operatorname{tr}(\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}) &= \operatorname{tr} I_T + 2(\rho_0 - \rho)\operatorname{tr} L_0 + (\rho_0 - \rho)^2 \operatorname{tr}(L_0 L_0') \\ &= T + (\rho_0 - \rho)^2 \operatorname{tr}(L_0 L_0'). \end{aligned} \tag{E.71}$$

Application of Lemma E.1 (a), $\exp(x) = 1 + x + O(x^2)$, and the fact that $\alpha = O(1)$, now yields

$$\begin{aligned}\operatorname{tr}(\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}) &= T + (\rho_0 - \rho)^2 \operatorname{tr}(L_0 L_0') \\ &= T + (c_0 - c)^2 \alpha^2 T^{-2} \operatorname{tr}(L_0 L_0') + O(T^{-1}) \\ &= T + (c_0 - c)^2 \alpha^2 h_0 + O(T^{-1}), \end{aligned} \tag{E.72}$$

which holds uniformly in c.

Next, consider $\operatorname{tr}(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1})$. Application of $\Gamma^{-1}\Gamma_0 = I_T + (\rho_0 - \rho)L_0$ yields

$$\begin{aligned}&\operatorname{tr}(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}) \\ &= \operatorname{tr}(D'\Gamma^{-1\prime}[I_T + (\rho_0 - \rho)L_0][I_T + (\rho_0 - \rho)L_0]'\Gamma^{-1}DQ^{-1}) \\ &= \operatorname{tr}(QQ^{-1}) + 2(\rho_0 - \rho)\operatorname{tr}(D'\Gamma^{-1\prime}L_0\Gamma^{-1}DQ^{-1}) + (\rho_0 - \rho)^2 \operatorname{tr}(D'\Gamma^{-1\prime}L_0L_0'\Gamma^{-1}DQ^{-1}) \\ &= 2 + 2(\rho_0 - \rho)\operatorname{tr}(D'\Gamma^{-1\prime}L_0\Gamma^{-1}DQ^{-1}) + (\rho_0 - \rho)^2 \operatorname{tr}(D'\Gamma^{-1\prime}L_0L_0'\Gamma^{-1}DQ^{-1}), \end{aligned} \tag{E.73}$$

According to Lemma E.1 (b), $\|N_T Q N_T - \overline{Q}\| = O(T^{-1/2})$, where $\overline{Q} = \operatorname{diag}[1, h_1(c)]$. Note how $h_1(c) > 0$ for all real triplets (α, c_0, c), implying $\overline{Q}$ is positive definite with inverse

$$\overline{Q}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/h_1(c) \end{bmatrix}.$$

Hence, by using the results of Andrews (1987), we can show that

$$(N_T Q N_T)^{-1} = \overline{Q}^{-1} + O(T^{-1/2}), \tag{E.74}$$

uniformly in c. By using this and Lemma E.1, we obtain

$$
\begin{aligned}
& N_T D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T(N_TQN_T)^{-1} \\
= \;& I_2 + 2(\rho_0-\rho)N_TD'\Gamma^{-1\prime}L_0\Gamma^{-1}DN_T(N_TQN_T)^{-1} \\
+ \;& (\rho_0-\rho)^2N_TD'\Gamma^{-1\prime}L_0L_0'\Gamma^{-1}DN_T(N_TQN_T)^{-1} \\
= \;& I_2 + 2(c_0-c)\alpha T^{-1}N_TD'\Gamma^{-1\prime}L_0\Gamma^{-1}DN_T(N_TQN_T)^{-1} \\
+ \;& (c_0-c)^2\alpha^2T^{-2}N_TD'\Gamma^{-1\prime}L_0L_0'\Gamma^{-1}DN_T(N_TQN_T)^{-1} + O(T^{-1}) \\
= \;& I_2 + 2(c_0-c)\alpha\begin{bmatrix} 0 & 0 \\ 0 & h_2(c)/h_1(c)\end{bmatrix} + (c_0-c)^2\alpha^2\begin{bmatrix} 0 & 0 \\ 0 & h_3(c)/h_1(c)\end{bmatrix} + O(T^{-1/2}) \\
= \;& \begin{bmatrix} 1 & 0 \\ 0 & 1+2(c_0-c)\alpha h_2(c)/h_1(c) + (c_0-c)^2\alpha^2h_3(c)/h_1(c)\end{bmatrix} + O(T^{-1/2}), \qquad \text{(E.75)}
\end{aligned}
$$

which again holds uniformly in c. This means that (E.73) can be written as follows:

$$
\begin{aligned}
& \operatorname{tr}(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}) \\
= \;& 2 + 2(c_0-c)\alpha h_2(c)/h_1(c) + (c_0-c)^2\alpha^2h_3(c)/h_1(c) + O(T^{-1/2}). \qquad \text{(E.76)}
\end{aligned}
$$

Hence, in view of the result previously obtained for $\operatorname{tr}(\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime})$ (with $D_t = (1,t)'$), we obtain

$$
\begin{aligned}
& \operatorname{tr}(\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}) - \operatorname{tr}(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}) \\
= \;& T - 2 - 2(c_0-c)\alpha h_2(c)/h_1(c) + (c_0-c)^2\alpha^2[h_0 - h_3(c)/h_1(c)] + O(T^{-1/2}) \\
= \;& T - 2 - g(c)/h_1(c) + O(T^{-1/2}), \qquad \text{(E.77)}
\end{aligned}
$$

uniformly in c, where

$$
\begin{aligned}
g(c) &= 2(c_0-c)\alpha h_2(c) - (c_0-c)^2\alpha^2[h_0h_1(c) - h_3(c)] \\
&= (c_0-c)\alpha g_1 + (c_0-c)^2\alpha^2 g_2 + (c_0-c)^3\alpha^3 g_3 + (c_0-c)^4\alpha^4 g_4,
\end{aligned}
$$

with

$$
\begin{aligned}
g_1 &= 2h_{21}, \\
g_2 &= 2h_{22} - h_{11}h_0 + h_{31}, \\
g_3 &= 2h_{23} - h_0h_{12} + h_{32}, \\
g_4 &= h_{33} - h_0h_{13}.
\end{aligned}
$$

This accounts for two of the terms of Q^* in (E.70).

Let us now consider $\log(|\hat{\Lambda}|)$. By Sylvester's determinant theorem, $|I_p + AB| = |I_q + BA|$ for any $p \times q$ matrix A and $q \times p$ matrix B (see Andrews, 1987, Exercise 5.37). This implies

$$\begin{aligned}\log(|\hat{\Lambda}|) &= \log(|I_T + \sigma^{-2}\Gamma^{-1}D\hat{S}_\lambda D'\Gamma^{-1\prime}|) = \log(|I_2 + \sigma^{-2}\hat{S}_\lambda D'\Gamma^{-1\prime}\Gamma^{-1}D|) \\ &= \log(|I_2 + \sigma^{-2}\hat{S}_\lambda Q|). \end{aligned} \tag{E.78}$$

Consider $\hat{S}_\lambda$. From

$$\begin{aligned} & Q^{-1}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1} \\ &= \sigma_0^2 Q^{-1}D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1} + S_\lambda + \frac{1}{N}\sum_{i=1}^N \lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1} \\ &+ Q^{-1}D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^N \varepsilon_i\lambda_i' + Q^{-1}D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}, \end{aligned} \tag{E.79}$$

we get, with $\sigma^2 = \sigma_0^2$,

$$\begin{aligned} \hat{S}_\lambda &= Q^{-1}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1} - \sigma^2 Q^{-1} \\ &= S_\lambda + \frac{1}{N}\sum_{i=1}^N \lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1} + Q^{-1}D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^N \varepsilon_i\lambda_i' \\ &+ Q^{-1}D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1} \\ &+ \sigma_0^2 Q^{-1}(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D - Q)Q^{-1} \\ &= S_\lambda + \frac{1}{N}\sum_{i=1}^N \lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T(N_TQN_T)^{-1}N_T \\ &+ N_T(N_TQN_T)^{-1}N_TD'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^N \varepsilon_i\lambda_i' \\ &+ N_T(N_TQN_T)^{-1}N_TD'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T(N_TQN_T)^{-1}N_T \\ &+ \sigma_0^2 N_T(N_TQN_T)^{-1}N_T(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D - Q)N_T(N_TQN_T)^{-1}N_T. \end{aligned} \tag{E.80}$$

Consider the second term on the right. By using the fact that $\Gamma^{-1}\Gamma_0 = I_T + (\rho_0 - \rho)L_0$, we have

$$\begin{aligned}
&\left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T\right\| \\
&= \left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'[I_T+(\rho_0-\rho)L_0]'\Gamma^{-1}DN_T\right\| \\
&\leq \left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T\right\| + |\rho_0-\rho|\left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'L_0'\Gamma^{-1}DN_T\right\| \\
&\leq N^{-1/2}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T\right\| \\
&+ N^{-1/2}T|\rho_0-\rho|\left\|\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\lambda_i\varepsilon_i'L_0'\Gamma^{-1}DN_T\right\|. \qquad \text{(E.81)}
\end{aligned}$$

Consider $N^{-1/2}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T$. By using $E(\varepsilon_i\varepsilon_i') = \sigma_0^2 I_T$, the cross-section independence of ε_i, the fact that $\varepsilon_i'\Gamma^{-1}DD'\Gamma^{-1\prime}\varepsilon_j$ is just a scalar, and $\|N_TQN_T\| = O(1)$, we get

$$\begin{aligned}
&E\left(\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T\right\|^2\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E[\mathrm{tr}(N_TD'\Gamma^{-1\prime}\varepsilon_i\lambda_i'\lambda_j\varepsilon_j'\Gamma^{-1}DN_T)] \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E(\varepsilon_j'\Gamma^{-1}DN_TN_TD'\Gamma^{-1\prime}\varepsilon_i)\mathrm{tr}(\lambda_j\lambda_i') \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathrm{tr}[E(\varepsilon_i\varepsilon_j')\Gamma^{-1}DN_TN_TD'\Gamma^{-1\prime}]\mathrm{tr}(\lambda_j\lambda_i') \\
&= \frac{1}{N}\sum_{i=1}^{N}\mathrm{tr}[E(\varepsilon_i\varepsilon_i')\Gamma^{-1}DN_TN_TD'\Gamma^{-1\prime}]\mathrm{tr}(\lambda_i\lambda_i') \\
&= \sigma_0^2\mathrm{tr}(N_TD'\Gamma^{-1\prime}\Gamma^{-1}DN_T)\mathrm{tr}(S_\lambda) = O(1),
\end{aligned}$$

and, by repeated use of the same argument,

$$E\left(\left\|\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\lambda_i\varepsilon_i'L_0'\Gamma^{-1}DN_T\right\|^2\right) = \sigma_0^2\mathrm{tr}(T^{-2}N_TD'\Gamma^{-1\prime}L_0L_0'\Gamma^{-1}DN_T)\mathrm{tr}(S_\lambda),$$

which is $O(1)$ for $\mathrm{tr}(T^{-2}N_TD'\Gamma^{-1\prime}L_0L_0'\Gamma^{-1}DN_T)$ is (see Lemma E.1). It follows that since $\|N^{-1/2}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T\|$ and $\|N^{-1/2}T^{-1}\sum_{i=1}^{N}\lambda_i\varepsilon_i'L_0'\Gamma^{-1}DN_T\|$ are both $O_p(1)$, and $T(\rho_0 -$

$\rho) = (c_0 - c)\alpha + O(T^{-1})$, we have

$$\begin{aligned}
&\left\| \frac{1}{N} \sum_{i=1}^{N} \lambda_i \varepsilon_i' \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D N_T \right\| \\
&\leq N^{-1/2} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_i \varepsilon_i' \Gamma^{-1} D N_T \right\| + N^{-1/2} T |\rho_0 - \rho| \left\| \frac{1}{\sqrt{N}T} \sum_{i=1}^{N} \lambda_i \varepsilon_i' L_0' \Gamma^{-1} D N_T \right\| \\
&= O_p(N^{-1/2}). \qquad \text{(E.82)}
\end{aligned}$$

We also know that $\|(N_T Q N_T)^{-1}\| = O(1)$ and $\|N_T\| = 1 + T^{-1/2}$, and so

$$\begin{aligned}
&\left\| \frac{1}{N} \sum_{i=1}^{N} \lambda_i \varepsilon_i' \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D N_T (N_T Q N_T)^{-1} N_T \right\| \\
&\leq N^{-1/2} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_i \varepsilon_i' \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D N_T \right\| \|(N_T Q N_T)^{-1}\| \|N_T\| \\
&= O_p(N^{-1/2}). \qquad \text{(E.83)}
\end{aligned}$$

The order of the third term on the right of (E.135) is the same.

Let us now consider the fourth term on the right-hand side of (E.135). Note how

$$\begin{aligned}
E(\|A(S_\varepsilon - \sigma_0^2 I_T)A'\|^2) &= E(\text{tr}[A(S_\varepsilon - \sigma_0^2 I_T)A'A(S_\varepsilon - \sigma_0^2 I_T)A']) \\
&= E(\text{tr}[AS_\varepsilon A'AS_\varepsilon A' - 2\sigma_0^2 AS_\varepsilon A'AA' + \sigma_0^4 AA'AA']) \\
&= \text{tr}[AE(S_\varepsilon A'AS_\varepsilon)A' - 2\sigma_0^2 AE(S_\varepsilon)A'AA' + \sigma_0^4 AA'AA'] \\
&= \text{tr}[E(AS_\varepsilon A'AS_\varepsilon A') - \sigma_0^4 AA'AA']
\end{aligned}$$

for any deterministic matrix A. Since $\sum_{i=2}^{N} = \sum_{i=1}^{N-1} = N(N-1)/2$ and $\text{tr}(A\varepsilon_i\varepsilon_i' A'A\varepsilon_i\varepsilon_i' A') = (\varepsilon_i' A'A\varepsilon_i)^2$, $\text{tr}[E(AS_\varepsilon A'AS_\varepsilon A')]$ can be written

$$\begin{aligned}
&\text{tr}[E(AS_\varepsilon A'AS_\varepsilon A')] \\
&= \frac{1}{N^2} \sum_{i=1}^{N} \text{tr}[E(A\varepsilon_i\varepsilon_i' A'A\varepsilon_i\varepsilon_i' A')] + \frac{2}{N^2} \sum_{i=2}^{N} \sum_{j=1}^{i-1} \text{tr}[AE(\varepsilon_i\varepsilon_i')A'AE(\varepsilon_j\varepsilon_j')A'] \\
&= \frac{1}{N^2} \sum_{i=1}^{N} E[(\varepsilon_i' A'A\varepsilon_i)^2] + N^{-1}(N-1)\sigma_0^4 \text{tr}(AA'AA'),
\end{aligned}$$

from which it follows that

$$E(\|A(S_\varepsilon - \sigma_0^2 I_T)A'\|^2) = \frac{1}{N^2} \sum_{i=1}^{N} E[(\varepsilon_i' A'A\varepsilon_i)^2] - N^{-1}\sigma_0^4 \text{tr}(AA'AA'). \qquad \text{(E.84)}$$

Now set $A = N_T D' \Gamma^{-1\prime} \Gamma^{-1} \Gamma_0$. We have shown that $\|N_T D' \Gamma^{-1\prime} \Gamma^{-1} \Gamma_0 \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D N_T\| = O(1)$. Therefore, $\text{tr}(AA'AA') = \|AA'\| = O(1)$. A tedious yet straightforward calculation reveals that $N^{-1}\sum_{i=1}^{N} E[(\varepsilon_i' A' A \varepsilon_i)^2] = O(1)$ (see Proof of Lemma E.2 for a similar calculation). It follows that

$$E(\|N_T D' \Gamma^{-1\prime} \Gamma^{-1} \Gamma_0 (S_\varepsilon - \sigma_0^2 I_T) \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D N_T\|^2) = O(N^{-1}),$$

and so

$$\|N_T D' \Gamma^{-1\prime} \Gamma^{-1} \Gamma_0 (S_\varepsilon - \sigma_0^2 I_T) \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D N_T\| = O_p(N^{-1/2}). \tag{E.85}$$

It remains to consider the last term on the right of (E.135). According to Lemma E.1,

$$\begin{aligned} & N_T D' \Gamma^{-1\prime} \Gamma^{-1} \Gamma_0 \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D N_T - N_T Q N_T \\ = \;& (c_0 - c)\alpha T^{-1} N_T D' \Gamma^{-1\prime} (L_0 + L_0') \Gamma^{-1} D N_T + (c_0 - c)^2 \alpha^2 T^{-2} N_T D' \Gamma^{-1\prime} L_0 L_0' \Gamma^{-1} D N_T \\ = \;& \begin{bmatrix} 0 & 0 \\ 0 & 2(c_0 - c)\alpha h_2(c) + (c_0 - c)^2 \alpha^2 h_3(c) \end{bmatrix} + O(T^{-1/2}). \end{aligned} \tag{E.86}$$

Hence, since $\|(N_T Q N_T)^{-1} - \overline{Q}^{-1}\| = O(T^{-1/2})$ with $\overline{Q}^{-1} = \mathrm{diag}[1, 1/h_1(c)]$,

$$\begin{aligned} & (N_T Q N_T)^{-1} N_T (D' \Gamma^{-1\prime} \Gamma^{-1} \Gamma_0 \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D - Q) N_T (N_T Q N_T)^{-1} \\ = \;& \begin{bmatrix} 0 & 0 \\ 0 & 2(c_0 - c)\alpha h_2(c)/h_1(c)^2 + (c_0 - c)^2 \alpha^2 h_3(c)/h_1(c)^2 \end{bmatrix} + O(T^{-1/2}), \end{aligned} \tag{E.87}$$

from which it follows that

$$\begin{aligned} & \|N_T (N_T Q N_T)^{-1} N_T (D' \Gamma^{-1\prime} \Gamma^{-1} \Gamma_0 \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D - Q) N_T (N_T Q N_T)^{-1} N_T\| \\ = \;& O(T^{-1/2}). \end{aligned} \tag{E.88}$$

Putting everything together, (E.135) reduces to

$$\begin{aligned} \hat{S}_\lambda \;=\;& S_\lambda + \frac{1}{N}\sum_{i=1}^{N} \lambda_i \varepsilon_i' \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D N_T (N_T Q N_T)^{-1} N_T \\ &+ N_T (N_T Q N_T)^{-1} N_T D' \Gamma^{-1\prime} \Gamma^{-1} \Gamma_0 \frac{1}{N}\sum_{i=1}^{N} \varepsilon_i \lambda_i' \\ &+ N_T (N_T Q N_T)^{-1} N_T D' \Gamma^{-1\prime} \Gamma^{-1} \Gamma_0 (S_\varepsilon - \sigma_0^2 I_T) \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D N_T (N_T Q N_T)^{-1} N_T \\ &+ \sigma_0^2 N_T (N_T Q N_T)^{-1} N_T (D' \Gamma^{-1\prime} \Gamma^{-1} \Gamma_0 \Gamma_0' \Gamma^{-1\prime} \Gamma^{-1} D - Q) N_T (N_T Q N_T)^{-1} N_T \\ =\;& S_\lambda + O_p(N^{-1/2}) + O_p(T^{-1/2}), \end{aligned} \tag{E.89}$$

which holds uniformly in c.

Let us now go back to (E.78). We had $\log(|\hat{\Lambda}|) = \log(|I_2 + \sigma^{-2}\hat{S}_\lambda Q|)$. From $|I_n + A| = 1 + |A| + \mathrm{tr}\, A$ and $|aA| = a^n |A|$ for any $n \times n$ matrix A (see Abadir and Magnus, 2005, Exercise 4.35),

$$\log(|\hat{\Lambda}|) = \log(|I_2 + \sigma^{-2}\hat{S}_\lambda Q|) = \log[1 + \sigma^{-4}|\hat{S}_\lambda Q| + \sigma^{-2}\mathrm{tr}\,(\hat{S}_\lambda Q)], \tag{E.90}$$

and by further use of $|AB| = |A||B|$ (see Abadir and Magnus, 2005, Exercise 4.42) and $|N_T^{-1}| = \sqrt{T}$, we obtain

$$\begin{aligned}\log(|\hat{\Lambda}|) &= \log[T^{-1} + \sigma^{-4}T^{-1}|\hat{S}_\lambda N_T^{-1}(N_T Q N_T)N_T^{-1}| + \sigma^{-2}T^{-1}\text{tr}\,(\hat{S}_\lambda Q)] + \log(T) \\ &= \log[T^{-1} + \sigma^{-4}T^{-1}|N_T^{-1}|^2|\hat{S}_\lambda||N_T Q N_T| + \sigma^{-2}T^{-1}\text{tr}\,(\hat{S}_\lambda Q)] + \log(T) \\ &= \log[T^{-1} + \sigma^{-4}|\hat{S}_\lambda||N_T Q N_T| + \sigma^{-2}\text{tr}\,(\hat{S}_\lambda T^{-1}Q)] + \log(T). \qquad \text{(E.91)}\end{aligned}$$

Let $S_{\lambda mn} = [S_\lambda]_{mn}$ be the element of S_λ that sits in row n and column m. Note how $S_{\lambda 12} = S_{\lambda 21}$. In this notation,

$$\begin{aligned}\text{tr}\,(\hat{S}_\lambda T^{-1}Q) &= \text{tr}\left(S_\lambda \begin{bmatrix} 0 & 0 \\ 0 & h_1(c) \end{bmatrix}\right) + O_p(N^{-1/2}) + O_p(T^{-1/2}) \\ &= S_{\lambda 22}h_1(c) + O_p(N^{-1/2}) + O_p(T^{-1/2}), \qquad \text{(E.92)}\end{aligned}$$

uniformly in c. From $|I_n + A| = 1 + |A| + \text{tr}\,A$, we have $|A + \varepsilon B| = |A||I_n + \varepsilon A^{-1}B| = |A|[1 + \varepsilon^n|A^{-1}B| + \varepsilon\,\text{tr}\,(A^{-1}B)] = |A| + O(\varepsilon)$ for $\varepsilon = o(1)$ and any $n \times n$ matrices A and B, where A is positive definite. This implies

$$|\hat{S}_\lambda| = |S_\lambda| + O_p(N^{-1/2}) + O_p(T^{-1/2}), \qquad \text{(E.93)}$$

$$|N_T Q N_T| = h_1(c) + O(T^{-1/2}). \qquad \text{(E.94)}$$

where $|S_\lambda| = S_{\lambda 11}S_{\lambda 22} - S_{\lambda 12}^2$. Insertion yields

$$\begin{aligned}&\log(|\hat{\Lambda}|) \\ &= \log[T^{-1} + \sigma^{-4}|\hat{S}_\lambda||N_T Q N_T| + \sigma^{-2}\text{tr}\,(\hat{S}_\lambda T^{-1}Q)] + \log(T) \\ &= \log[\sigma^{-4}(S_{\lambda 11}S_{\lambda 22} - S_{\lambda 12}^2)h_1(c) + \sigma^{-2}S_{\lambda 22}h_1(c)] + \log(T) + O_p(N^{-1/2}) + O_p(T^{-1/2}) \\ &= \log[\sigma^{-4}(S_{\lambda 11}S_{\lambda 22} - S_{\lambda 12}^2) + \sigma^{-2}S_{\lambda 22}] + \log[h_1(c)] + \log(T) \\ &+ O_p(N^{-1/2}) + O_p(T^{-1/2}), \qquad \text{(E.95)}\end{aligned}$$

uniformly in c.

The results used to obtain the above expression for $\log(|\hat{\Lambda}|)$ also imply

$$\begin{aligned}&\text{tr}[D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}] \\ &= \text{tr}[N_T D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T(N_T Q N_T)^{-1}] \\ &= O_p(N^{-1/2}). \qquad \text{(E.96)}\end{aligned}$$

We have now considered all terms in (E.70), except $\text{tr}\,[\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}]$. From $\Gamma^{-1}\Gamma_0 = I_T + (\rho_0 - \rho)L_0$,

$$\begin{aligned}
&\text{tr}\,[\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}] \\
&= \text{tr}\,[(I_T + (\rho_0 - \rho)L_0)(S_\varepsilon - \sigma_0^2 I_T)(I_T + (\rho_0 - \rho)L_0)'] \\
&= \text{tr}\,(S_\varepsilon - \sigma_0^2 I_T) + 2T(\rho_0 - \rho)T^{-1}\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)L_0] \\
&+ T^2(\rho_0 - \rho)^2 T^{-2}\text{tr}\,[L_0'(S_\varepsilon - \sigma_0^2 I_T)L_0]. \qquad \text{(E.97)}
\end{aligned}$$

The steps used for evaluating $E(\|N_T D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T\|^2)$ can be applied also to $E[(T^{-1}\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)L_0])^2]$, $E[(T^{-2}\text{tr}\,[L_0'(S_\varepsilon - \sigma_0^2 I_T)L_0])^2]$ and $E([\text{tr}\,(S_\varepsilon - \sigma_0^2 I_T)]^2)$, the first of which is given by

$$\begin{aligned}
E[(T^{-1}\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)L_0])^2] &= \frac{1}{(NT)^2}\sum_{i=1}^{N} E[(\varepsilon_i'\varepsilon_i)(\varepsilon_i' L_0 L_0' \varepsilon_i)] - N^{-1}T^{-2}\sigma_0^4\text{tr}(L_0 L_0') \\
&= O(N^{-1}).
\end{aligned}$$

Hence, $T^{-1}\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)L_0] = O_p(N^{-1/2})$ and we can similarly show that $T^{-2}\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)L_0 L_0']$ is of the same order. By using this and $T(\rho_0 - \rho) = (c_0 - c)\alpha + O(T^{-1})$, we obtain

$$\begin{aligned}
&\text{tr}\,[\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}] \\
&= \text{tr}\,(S_\varepsilon - \sigma_0^2 I_T) + 2T(\rho_0 - \rho)T^{-1}\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)L_0] + T^2(\rho_0 - \rho)^2 T^{-2}\text{tr}\,[L_0'(S_\varepsilon - \sigma_0^2 I_T)L_0] \\
&= \text{tr}\,(S_\varepsilon - \sigma_0^2 I_T) + 2(c_0 - c)\alpha T^{-1}\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)L_0] \\
&+ (c_0 - c)^2\alpha^2 T^{-2}\text{tr}\,[L_0'(S_\varepsilon - \sigma_0^2 I_T)L_0] + O_p(N^{-1/2}T^{-1}) \\
&= \text{tr}\,(S_\varepsilon - \sigma_0^2 I_T) + O_p(N^{-1/2}) + O_p(T^{-1}), \qquad \text{(E.98)}
\end{aligned}$$

uniformly in c.

We now have all the pieces needed to evaluate Q^*. Direct insertion into (E.70), and using

$\sigma^2 = \sigma_0^2$ and $D_t = (1, t)'$,

$$
\begin{aligned}
&Q^* \\
=\ & T\log(\sigma^2) + \log(|\hat{\Lambda}|) \\
+\ & \sigma^{-2}\sigma_0^2[\text{tr}\,(\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}) - \text{tr}\,(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1})] \\
+\ & \sigma^{-2}\text{tr}\,[\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}] \\
-\ & \sigma^{-2}\text{tr}\,[D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1}] + 2 \\
=\ & T\log(\sigma^2) + \log[\sigma^{-4}(S_{\lambda 11}S_{\lambda 22} - S_{\lambda 12}^2) + \sigma^{-2}S_{\lambda 22}] + \log[h_1(c)] + \log(T) \\
+\ & \sigma^{-2}\sigma_0^2(T-2) - \sigma^{-2}\sigma_0^2 g(c)/h_1(c) + \sigma^{-2}\text{tr}\,(S_\varepsilon - \sigma_0^2 I_T) + 2 + O_p(T^{-1/2}) + O_p(N^{-1/2}) \\
=\ & q(c) + T\log(\sigma_0^2) + \log[\sigma_0^{-4}(S_{\lambda 11}S_{\lambda 22} - S_{\lambda 12}^2) + \sigma_0^{-2}S_{\lambda 22}] + \log(T) + T \\
+\ & \sigma_0^{-2}\text{tr}\,(S_\varepsilon - \sigma_0^2 I_T) + O_p(T^{-1/2}) + O_p(N^{-1/2}),
\end{aligned}
\tag{E.99}
$$

where

$$q(c) = \log[h_1(c)] - \frac{g(c)}{h_1(c)}.$$

The order of the remainder is again uniform in c. Note also how $q(c)$ is everywhere differentiable, because $h_1(c) > 0$ for all c. It follows that

$$\frac{d}{dc}Q^* = \frac{d}{dc}q(c) + O_p(T^{-1/2}) + O_p(N^{-1/2}). \tag{E.100}$$

Consider $dq(c)/dc$. Clearly,

$$\frac{d}{dc}h_1(c) = -[\alpha h_{12} + 2(c_0 - c)\alpha^2 h_{13}], \tag{E.101}$$

$$\frac{d}{dc}h_2(c) = -[\alpha h_{22} + 2(c_0 - c)\alpha^2 h_{23}], \tag{E.102}$$

$$\frac{d}{dc}g(c) = -[\alpha g_1 + 2(c_0 - c)\alpha^2 g_2 + 3(c_0 - c)^2\alpha^3 g_3 + 4(c_0 - c)^3\alpha^4 g_4]. \tag{E.103}$$

These results yield, after considerable simplification,

$$
\begin{aligned}
\frac{d}{dc}q(c) &= \frac{d}{dc}\left(\log[h_1(c)] - \frac{g(c)}{h_1(c)}\right) \\
&= \frac{1}{h_1(c)}\frac{d}{dc}h_1(c) - \frac{1}{h_1(c)^2}\left(h_1(c)\frac{d}{dc}g(c) - g(c)\frac{d}{dc}h_1(c)\right) \\
&= -\frac{1}{h_1(c)^2}(h_1(c)[\alpha h_{12} + 2(c_0 - c)\alpha^2 h_{13}] \\
&- h_1(c)[\alpha g_1 + 2(c_0 - c)\alpha^2 g_2 + 3(c_0 - c)^2\alpha^3 g_3 + 4(c_0 - c)^3\alpha^4 g_4] \\
&+ g(c)[\alpha h_{12} + 2(c_0 - c)\alpha^2 h_{13}]) \\
&= -\frac{1}{h_1(c)^2}[\alpha q_1 + (c_0 - c)\alpha^2 q_2 + (c_0 - c)^2\alpha^3 q_3 + (c_0 - c)^3\alpha^4 q_4 \\
&+ (c_0 - c)^4\alpha^5 q_5 + (c_0 - c)^5\alpha^6 q_6]. \qquad \text{(E.104)}
\end{aligned}
$$

where

$$
\begin{aligned}
q_1 &= h_{11}h_{12} - h_{11}g_1, \\
q_2 &= 2h_{11}h_{13} + h_{12}^2 - 2h_{11}g_2 - h_{12}g_1 + 2h_{21}h_{12}, \\
q_3 &= 3h_{13}h_{12} - 3h_{11}g_3 - 2h_{12}g_2 - h_{13}g_1 + 2h_{22}h_{12} + 4h_{21}h_{13} \\
&- h_0h_{11}h_{12} + h_{12}h_{31}, \\
q_4 &= 2h_{13}^2 - 4h_{11}g_4 - 3h_{12}g_3 - 2h_{13}g_2 + 2h_{23}h_{12} + 4h_{22}h_{13} - h_0h_{12}^2, \\
&- 2h_0h_{11}h_{13} + 2h_{13}h_{31} + 2h_{32}h_{12}, \\
q_5 &= -4h_{12}g_4 - 3h_{13}g_3 + 4h_{23}h_{13} - h_0h_{13}h_{12} - 2h_0h_{12}h_{13} \\
&+ 4h_{32}h_{13} + h_{33}h_{12}, \\
q_6 &= -4h_{13}g_4 - 2h_0h_{13}^2 + 2h_{33}h_{13}.
\end{aligned}
$$

The proof is completed by noting that

$$
\begin{aligned}
q_1 &= h_{11}(h_{12} - g_1) = h_{11}(h_{12} - 2h_{21}) \\
&= h_{11}[(1 - 2c_0\alpha/3) - 2(1/2 - c_0\alpha/3)] = 0. \qquad \text{(E.105)}
\end{aligned}
$$

■

Lemma E.2. *Suppose that Assumptions 1–5 hold, and $D_t = (1, t)'$ with $\alpha_0 \in \mathbb{A} \setminus \{0\}$ and $c_0 \in$*

$\mathbb{C} \setminus \{0\}$. *Then, as* N, $T \to \infty$,

(a) $$\frac{1}{\sqrt{NT}} \frac{\partial \ell^*(\rho_0)}{\partial \rho} \to_d N\left(0, \lim_{N,T\to\infty} s_0^2\right),$$

(b) $$\frac{1}{\sqrt{NT}} \frac{\partial \ell^*}{\partial \sigma^2} \to_d N\left(0, \frac{(\kappa_0 - 1)}{4\sigma_0^4}\right),$$

(c) $$E\left(\frac{1}{NT^{3/2}} \frac{\partial \ell^*}{\partial \sigma^2} \frac{\partial \ell^*}{\partial \rho}\right) = o(1),$$

(d) $$\frac{1}{NT^2} \frac{\partial^2 \ell^*(\rho_0)}{(\partial \rho)^2} \to_p - \lim_{N,T\to\infty} s_0^2,$$

(e) $$\frac{1}{NT^{3/2}} \frac{\partial^2 \ell^*}{\partial \rho \partial \sigma^2} = o_p(1),$$

(f) $$\frac{1}{NT} \frac{\partial^2 \ell^*}{(\partial \sigma^2)^2} \to_p -\frac{1}{2\sigma_0^4},$$

where $s_0^2 = s^2(\rho_0) = T^{-2}\mathrm{tr}\,[(L(\rho_0)' + L(\rho_0))M_{\Gamma(\rho_0)^{-1}D} L(\rho_0) M_{\Gamma(\rho_0)^{-1}D}]$.

Proof: In this proof we set $\theta_2 = \theta_2^0$, and use Γ_0 and L_0 to denote the true values of Γ and L, respectively.

Consider (a). We begin by considering the most general case when $D_t = (1, t)'$. Consider $\partial \ell^* / \partial \rho$. From $G = \Gamma_0^{-1} S_y \Gamma_0^{-1\prime} = S_u$,

$$\frac{\sigma_0^2}{\sqrt{NT}} \frac{\partial \ell^*}{\partial \rho} = \sqrt{N} T^{-1}(R_1 + R_2), \tag{E.106}$$

where

$$R_1 = \mathrm{tr}\,(M_{\Gamma_0^{-1}D} S_u M_{\Gamma_0^{-1}D} L_0 - \sigma_0^2 L_0 M_{\Gamma_0^{-1}D}),$$
$$R_2 = \mathrm{tr}\,[\sigma_0^2 (D'\Gamma_0^{-1\prime} S_u \Gamma_0^{-1} D)^{-1} D' \Gamma_0^{-1\prime} (L_0 + L_0') M_{\Gamma_0^{-1}D} S_u \Gamma_0^{-1} D].$$

Consider R_1. Using $M_{\Gamma_0^{-1}D} \Gamma_0^{-1} D = 0_{T\times 2}$ and the definition of S_u, we get

$$\begin{aligned} M_{\Gamma_0^{-1}D} S_u M_{\Gamma_0^{-1}D} &= M_{\Gamma_0^{-1}D} \frac{1}{N} \sum_{i=1}^N (\Gamma_0^{-1} D\lambda_i + \varepsilon_i)(\Gamma_0^{-1} D\lambda_i + \varepsilon_i)' M_{\Gamma_0^{-1}D} \\ &= M_{\Gamma_0^{-1}D} \Gamma_0^{-1} D S_\lambda D' \Gamma_0^{-1\prime} M_{\Gamma_0^{-1}D} + \frac{1}{N} \sum_{i=1}^N M_{\Gamma_0^{-1}D} \Gamma_0^{-1} D \lambda_i \varepsilon_i' M_{\Gamma_0^{-1}D} \\ &+ \frac{1}{N} \sum_{i=1}^N M_{\Gamma_0^{-1}D} \varepsilon_i \lambda_i' D' \Gamma_0^{-1\prime} M_{\Gamma_0^{-1}D} + \frac{1}{N} \sum_{i=1}^N M_{\Gamma_0^{-1}D} \varepsilon_i \varepsilon_i' M_{\Gamma_0^{-1}D} \\ &= \frac{1}{N} \sum_{i=1}^N M_{\Gamma_0^{-1}D} \varepsilon_i \varepsilon_i' M_{\Gamma_0^{-1}D}, \end{aligned} \tag{E.107}$$

so that

$$\begin{aligned}
R_1 &= \mathrm{tr}\,(M_{\Gamma_0^{-1}D}S_u M_{\Gamma_0^{-1}D}L_0 - \sigma_0^2 L_0 M_{\Gamma_0^{-1}D}) \\
&= \frac{1}{N}\sum_{i=1}^{N}\mathrm{tr}\,(M_{\Gamma_0^{-1}D}\varepsilon_i\varepsilon_i' M_{\Gamma_0^{-1}D}L_0 - \sigma_0^2 M_{\Gamma_0^{-1}D}L_0) \\
&= \frac{1}{N}\sum_{i=1}^{N}[\varepsilon_i' M_{\Gamma_0^{-1}D}L_0 M_{\Gamma_0^{-1}D}\varepsilon_i - \sigma_0^2\mathrm{tr}\,(M_{\Gamma_0^{-1}D}L_0 M_{\Gamma_0^{-1}D})] \\
&= \frac{1}{N}\sum_{i=1}^{N}(\varepsilon_i' A\varepsilon_i - \sigma_0^2\mathrm{tr}\,A), \qquad \text{(E.108)}
\end{aligned}$$

where $A = M_{\Gamma_0^{-1}D}L_0 M_{\Gamma_0^{-1}D}$. This term is clearly mean zero. For the variance, note how $\varepsilon_i' A\varepsilon_i = \sum_{t=1}^{T}\sum_{s=1}^{T}\varepsilon_{i,t}\varepsilon_{i,s}a_{ts}$, where $a_{nm} = [A]_{nm}$ is the element of A that sits in row n and column m. Recall that $\kappa_0 = E(\varepsilon_{i,t}^4)/\sigma_0^4$. Since $E(\varepsilon_i' A\varepsilon_i) = \mathrm{tr}\,[E(\varepsilon_i\varepsilon_i')A] = \sigma_0^2\mathrm{tr}\,A$, $\mathrm{tr}\,A = \sum_{t=1}^{T}a_{tt}$ and

$$\begin{aligned}
E(\varepsilon_i' A\varepsilon_i\varepsilon_i' A\varepsilon_i) &= \sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{k=1}^{T}\sum_{n=1}^{T}E(\varepsilon_{i,t}\varepsilon_{i,s}\varepsilon_{i,k}\varepsilon_{i,n})a_{ts}a_{kn} \\
&= \sum_{t=1}^{T}E(\varepsilon_{i,t}^4)a_{tt}^2 + \sum_{t=1}^{T}\sum_{n=1}^{t-1}E(\varepsilon_{i,t}^2)E(\varepsilon_{i,n}^2)a_{tt}a_{nn} + \sum_{t=1}^{T}\sum_{n=t+1}^{T}E(\varepsilon_{i,t}^2)E(\varepsilon_{i,n}^2)a_{tt}a_{nn} \\
&+ \sum_{t=1}^{T}\sum_{n=1}^{t-1}E(\varepsilon_{i,t}^2)E(\varepsilon_{i,n}^2)a_{tn}^2 + \sum_{t=1}^{T}\sum_{n=t+1}^{T}E(\varepsilon_{i,t}^2)E(\varepsilon_{i,n}^2)a_{tn}^2 \\
&+ \sum_{t=1}^{T}\sum_{s=1}^{t-1}E(\varepsilon_{i,t}^2)E(\varepsilon_{i,s}^2)a_{ts}a_{st} + \sum_{t=1}^{T}\sum_{s=t+1}^{T}E(\varepsilon_{i,t}^2)E(\varepsilon_{i,s}^2)a_{ts}a_{st} \\
&= \sigma_0^4(\kappa_0-3)\sum_{t=1}^{T}a_{tt}^2 + \sigma_0^4\sum_{t=1}^{T}\sum_{n=1}^{T}a_{tt}a_{nn} + \sigma_0^4\sum_{t=1}^{T}\sum_{n=1}^{T}a_{tn}^2 + \sigma_0^4\sum_{t=1}^{T}\sum_{n=1}^{T}a_{tn}a_{nt},
\end{aligned}$$

we obtain

$$\begin{aligned}
&E(\mathrm{tr}\,[M_{\Gamma_0^{-1}D}\varepsilon_i\varepsilon_i' M_{\Gamma_0^{-1}D}L_0 - \sigma_0^2 M_{\Gamma_0^{-1}D}M_{\Gamma_0^{-1}D}L_0]^2) \\
&= E[(\varepsilon_i' A\varepsilon_i - \sigma_0^2\mathrm{tr}\,A)^2] \\
&= E(\varepsilon_i' A\varepsilon_i\varepsilon_i' A\varepsilon_i) - 2\sigma_0^2 E(\varepsilon_i' A\varepsilon_i)\mathrm{tr}\,A + \sigma_0^4(\mathrm{tr}\,A)^2 \\
&= E(\varepsilon_i' A\varepsilon_i\varepsilon_i' A\varepsilon_i) - \sigma_0^4(\mathrm{tr}\,A)^2 \\
&= \sigma_0^4(\kappa_0-3)\sum_{t=1}^{T}a_{tt}^2 + \sigma_0^4\sum_{t=1}^{T}\sum_{n=1}^{T}a_{tt}a_{nn} + \sigma_0^4\sum_{t=1}^{T}\sum_{n=1}^{T}a_{tn}^2 + \sigma_0^4\sum_{t=1}^{T}\sum_{n=1}^{T}a_{tn}a_{nt} - \sigma_0^4\sum_{t=1}^{T}\sum_{n=1}^{T}a_{tt}a_{nn} \\
&= \sigma_0^4(\kappa_0-3)\sum_{t=1}^{T}a_{tt}^2 + \sigma_0^4\sum_{t=1}^{T}\sum_{n=1}^{T}a_{tn}^2 + \sigma_0^4\sum_{t=1}^{T}\sum_{n=1}^{T}a_{tn}a_{nt} \\
&= \sigma_0^4(\kappa_0-3)\mathrm{tr}\,(A\circ A) + \sigma_0^4\mathrm{tr}\,(A'A) + \sigma_0^4\mathrm{tr}\,(AA), \qquad \text{(E.109)}
\end{aligned}$$

where $\circ$ signifies element wise (Hadamard) multiplication. Here,

$$\begin{aligned}
T^{-2}\text{tr}\,(A'A) &= T^{-2}\text{tr}\,(M_{\Gamma_0^{-1}D}L_0'M_{\Gamma_0^{-1}D}L_0M_{\Gamma_0^{-1}D}) \\
&= T^{-2}\text{tr}\,[(I_T - P_{\Gamma_0^{-1}D})L_0'(I_T - P_{\Gamma_0^{-1}D})L_0(I_T - P_{\Gamma_0^{-1}D})] \\
&= T^{-2}\text{tr}\,(L_0'L_0 - L_0'L_0P_{\Gamma_0^{-1}D} - L_0'P_{\Gamma_0^{-1}D}L_0 + L_0'P_{\Gamma_0^{-1}D}L_0P_{\Gamma_0^{-1}D} \\
&- P_{\Gamma_0^{-1}D}L_0'L_0 + P_{\Gamma_0^{-1}D}L_0'L_0P_{\Gamma_0^{-1}D} + P_{\Gamma_0^{-1}D}L_0'P_{\Gamma_0^{-1}D}L_0 - P_{\Gamma_0^{-1}D}L_0'P_{\Gamma_0^{-1}D}L_0P_{\Gamma_0^{-1}D}) \\
&= T^{-2}\text{tr}\,(L_0'L_0 - L_0'L_0P_{\Gamma_0^{-1}D} - L_0L_0'P_{\Gamma_0^{-1}D} + L_0'P_{\Gamma_0^{-1}D}L_0P_{\Gamma_0^{-1}D}), \\
T^{-2}\text{tr}\,(AA) &= T^{-2}\text{tr}\,(L_0L_0 - 2L_0L_0P_{\Gamma_0^{-1}D} + L_0P_{\Gamma_0^{-1}D}L_0P_{\Gamma_0^{-1}D}).
\end{aligned}$$

Moreover, $\text{tr}\,(A \circ A)$ is dominated by $\text{tr}\,(A'A)$ and $\text{tr}\,(AA)$. In fact, a direct calculation reveals that $\text{tr}\,(A'A) = O(T)$, which in turn implies

$$\begin{aligned}
& T^{-2}E(\text{tr}\,[M_{\Gamma_0^{-1}D}\varepsilon_i\varepsilon_i'M_{\Gamma_0^{-1}D}L_0 - \sigma_0^2 M_{\Gamma_0^{-1}D}M_{\Gamma_0^{-1}D}L_0]^2) \\
&= \sigma_0^4 T^{-2}[\text{tr}\,(A'A) + \text{tr}\,(AA)] + O(T^{-1}) \\
&= \sigma_0^4 T^{-2}\text{tr}\,[(L_0' + L_0)M_{\Gamma_0^{-1}D}L_0M_{\Gamma_0^{-1}D}] + O(T^{-1}) \\
&= \sigma_0^4 T^{-2}\text{tr}\,(L_0L_0') - \sigma_0^4\text{tr}\,[T^{-2}D'\Gamma_0^{-1\prime}(L_0'L_0 + L_0L_0)\Gamma_0^{-1}DN_T(N_TQN_T)^{-1}] \\
&- \sigma_0^4\text{tr}\,[T^{-2}N_TD'\Gamma_0^{-1\prime}(L_0L_0' + L_0L_0)\Gamma_0^{-1}DN_T(N_TQN_T)^{-1}] \\
&+ \sigma_0^4\text{tr}\,[T^{-1}N_TD'\Gamma_0^{-1\prime}(L_0' + L_0)\Gamma_0^{-1}DN_T(N_TQN_T)^{-1} \\
&\times T^{-1}N_TD'\Gamma_0^{-1\prime}L_0\Gamma_0^{-1}DN_T(N_TQN_T)^{-1}] + O(T^{-1}). \qquad \text{(E.110)}
\end{aligned}$$

We know from Lemma E.1 and Proof of Lemma E.2 that

$$\begin{aligned}
T^{-2}\text{tr}\,(L_0L_0') &= h_0 + O(T^{-1}), \\
(N_TQN_T)^{-1} &= \overline{Q}^{-1} + O(T^{-1/2}), \\
T^{-1}N_TD'\Gamma_0^{-1\prime}L_0\Gamma_0^{-1}DN_T &= \begin{bmatrix} 0 & 0 \\ 0 & h_2(c) \end{bmatrix} + O(T^{-1/2}),
\end{aligned}$$

and by the asymptotic symmetry of $T^{-1}N_TD'\Gamma_0^{-1\prime}L_0\Gamma_0^{-1}DN_T$, we also have

$$T^{-1}N_TD'\Gamma_0^{-1\prime}(L_0 + L_0')\Gamma_0^{-1}DN_T = \begin{bmatrix} 0 & 0 \\ 0 & 2h_2(c) \end{bmatrix} + O(T^{-1/2}). \qquad \text{(E.111)}$$

Also, using again the results of Lemma E.1,

$$\begin{aligned}
T^{-2}N_TD'\Gamma_0^{-1\prime}(L_0L_0' + L_0L_0)\Gamma_0^{-1}DN_T &= \begin{bmatrix} 0 & 0 \\ 0 & h_3(c) + h_4(c) \end{bmatrix} + O(T^{-1/2}), \qquad \text{(E.112)} \\
T^{-2}N_TD'\Gamma_0^{-1\prime}(L_0'L_0 + L_0L_0)\Gamma_0^{-1}DN_T &= \begin{bmatrix} 0 & 0 \\ 0 & h_4(c) + h_5(c) \end{bmatrix} + O(T^{-1/2}). \qquad \text{(E.113)}
\end{aligned}$$

Insertion and simplification yield

$$
\begin{aligned}
& T^{-2}\mathrm{tr}\,[(L_0' + L_0)M_{\Gamma_0^{-1}D}L_0M_{\Gamma_0^{-1}D}] \\
= \;& h_0 - \mathrm{tr}\begin{bmatrix} 0 & 0 \\ 0 & (h_4(c)+h_5(c))/h_1(c) \end{bmatrix} - \mathrm{tr}\begin{bmatrix} 0 & 0 \\ 0 & (h_3(c)+h_4(c))/h_1(c) \end{bmatrix} \\
+ \;& \mathrm{tr}\begin{bmatrix} 0 & 0 \\ 0 & 2h_2(c)^2/h_1(c)^2 \end{bmatrix} + O(T^{-1/2}) \\
= \;& \omega^2(c) + O(T^{-1/2}),
\end{aligned} \tag{E.114}
$$

where

$$
\omega^2(c) = h_0 - [h_3(c) + 2h_4(c) + h_5(c)]/h_1(c) + 2h_2(c)^2/h_1(c)^2,
$$

which is the limiting representation of $s^2(\rho)$. Note that this representation only applies in the $D_t = (1, t)'$ case. The term we seek is $\omega_0^2 = \omega^2(c_0)$, which is given by

$$
\omega_0^2 = h_0 - (h_{31} + 2h_{41} + h_{51})/h_{11} + 2h_{21}^2/h_{11}^2. \tag{E.115}
$$

Further use of the definitions of h_0, h_{11}, h_{21}, h_{31}, h_{41} and h_{51}, and $\exp(x) = \sum_{j=0}^{\infty} x^j/j!$ yields

$$
\begin{aligned}
& \omega_0^2 \\
= \;& \frac{1}{[4\alpha^3c_0^3(3 - 3\alpha c_0 + \alpha^2c_0^2)^2]}[72 - 9\alpha c_0 - 30\alpha^2c_0^2 - 3\alpha^3c_0^3 + 6\alpha^4c_0^4 + 3\alpha^5c_0^5 - 2\alpha^6c_0^6 \\
+ \;& (-72 + 153\alpha c_0 - 132\alpha^2c_0^2 + 57\alpha^3c_0^3 - 12\alpha^4c_0^4 + \alpha^5c_0^5)\phi_0(2)] \\
= \;& \frac{\alpha^2c_0^2}{[6(3 - 3\alpha c_0 + \alpha^2c_0^2)^2]}\Bigg(30 - 54\alpha c_0 + 36\alpha^2c_0^2 - 10\alpha^3c_0^3 + \alpha^4c_0^4 \\
+ \;& 48(-72 + 153\alpha c_0 - 132\alpha^2c_0^2 + 57\alpha^3c_0^3 - 12\alpha^4c_0^4 + \alpha^5c_0^5)\sum_{j=0}^{\infty}\frac{(2\alpha c_0)^j}{(j+5)!}\Bigg) \\
= \;& \frac{\alpha^2c_0^2}{[30(3 - 3\alpha c_0 + \alpha^2c_0^2)^2]}\Bigg(6 + 36\alpha c_0 - 84\alpha^2c_0^2 + 64\alpha^3c_0^3 - 19\alpha^4c_0^4 + 2\alpha^5c_0^5 \\
+ \;& 240(-72 + 153\alpha c_0 - 132\alpha^2c_0^2 + 57\alpha^3c_0^3 - 12\alpha^4c_0^4 + \alpha^5c_0^5)\sum_{j=0}^{\infty}\frac{(2\alpha c_0)^{j+1}}{(j+6)!}\Bigg).
\end{aligned} \tag{E.116}
$$

It follows that

$$
\begin{aligned}
& E[(\sqrt{N}T^{-1}R_1^2) \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E(\text{tr}\,[M_{\Gamma_0^{-1}D}\varepsilon_i\varepsilon_j' M_{\Gamma_0^{-1}D}L_0 - \sigma_0^2 M_{\Gamma_0^{-1}D}M_{\Gamma_0^{-1}D}L_0]^2) \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E[(\varepsilon_i' A\varepsilon_i - \sigma_0^2\text{tr}\,A)(\varepsilon_j' A\varepsilon_j - \sigma_0^2\text{tr}\,A)] \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}E[(\varepsilon_i' A\varepsilon_i - \sigma_0^2\text{tr}\,A)^2] + \frac{2}{NT^2}\sum_{i=2}^{N}\sum_{j=1}^{i-1}E(\varepsilon_i' A\varepsilon_i - \sigma_0^2\text{tr}\,A)E(\varepsilon_j' A\varepsilon_j - \sigma_0^2\text{tr}\,A) \\
= & \frac{1}{NT^2}\sum_{i=1}^{N}E[(\varepsilon_i' A\varepsilon_i - \sigma_0^2\text{tr}\,A)^2] = \sigma_0^4\omega_0^2 + O(T^{-1/2}).
\end{aligned} \tag{E.117}
$$

We want to show that $\sqrt{N}T^{-1}R_1$ converges to a normal variate. This is accomplished by applying Lindeberg central limit theorem for the joint N, T expansion given in Theorem 2 in Phillips and Moon (1999). In the notation of Phillips and Moon (1999), we have $\xi_{i,N,T} = T^{-1}(\varepsilon_i' A\varepsilon_i - \sigma_0^2\text{tr}A)$. The Lindeberg condition for this variable is given by

$$
\frac{1}{N}\sum_{i=1}^{N}E\left(\xi_{i,N,T}^2\right)1\left(\xi_{i,N,T}^2 > N\varepsilon\right) = o(1) \tag{E.118}
$$

as $N, T \to \infty$. Observe that $\xi_{i,N,T}$ is a function of N, as well. This is so, because it depends on ρ_0 and thus on $\alpha = \alpha(N, T) = O(1)$. Therefore, the convergence explored below holds for large N and T. Since $\xi_{i,N,T}$ is iid across i, the condition simplifies to $E(\xi_{1,N,T}^2)1(\xi_{1,N,T}^2 > N\varepsilon) = o(1)$, which holds if $\xi_{1,N,T}^2$ uniformly integrable over T. Because $\xi_{1,N,T}^2 \geq 0$, uniform integrability is equivalent to requiring (i) $\xi_{1,N,T} \to_d \xi_1$ and (ii) $E(\xi_{1,N,T}^2) \to E(\xi_1^2)$ (see Moon and Phillips, 2000, page 791). We start by verifying (i). Let $d = (d_1, ..., d_T)' = \Gamma_0^{-1}D = (I_T - \rho_0 J)(1_T, t_T) = (1_T - \rho_0 E_1, t_T - \rho_0 J t_T)$, where $d_t = (1, 1)'$ for $t = 1$ and $d_t = [1 - \rho_0, t - \rho_0(t-1)]$ for $t \geq 2$. Let us also introduce $\varepsilon_i^* = M_{\Gamma_0^{-1}D}\varepsilon_i$, the t-th element of which is given by $\varepsilon_{i,t}^* = \varepsilon_{i,t} - \hat{\delta}' d_t$, where $\hat{\delta} = Q^{-1}D'\Gamma_0^{-1\prime}\varepsilon_i$. Note how

$$
N_T^{-1}\hat{\delta} = (N_T Q N_T)^{-1} N_T D'\Gamma_0^{-1\prime}\varepsilon_i = (N_T Q N_T)^{-1}\left[\begin{array}{c} (1_T - \rho_0 E_1)'\varepsilon_i \\ T^{-1/2}(t_T - \rho_0 J t_T)'\varepsilon_i \end{array}\right],
$$

where

$$
\begin{aligned}
(1_T - \rho_0 E_1)'\varepsilon_i &= (1_T - E_1)'\varepsilon_i - (\rho_0 - 1)E_1'\varepsilon_i \\
&= (1_T - E_1)'\varepsilon_i - c_0\alpha T^{-1}E_1'\varepsilon_i + O_p(T^{-3/2}) \\
&= (1_T - E_1)'\varepsilon_i + O_p(T^{-1/2}) = \varepsilon_{i,1} + O_p(T^{-1/2}), \\
T^{-1/2}(t_T - \rho_0 J t_T)'\varepsilon_i &= T^{-1/2}(t_T - Jt_T)'\varepsilon_i - T^{-1/2}(\rho_0 - 1)(Jt_T)'\varepsilon_i \\
&= T^{-1/2}[(I_T - J)t_T]'\varepsilon_i - c_0\alpha T^{-3/2}(Jt_T)'\varepsilon_i + O_p(T^{-1}) \\
&= T^{-1/2}1_T'\varepsilon_i - c_0\alpha T^{-3/2}t_T'J'\varepsilon_i + O_p(T^{-1}) \\
&= T^{-1/2}\sum_{t=1}^{T}\varepsilon_{i,t} - c_0\alpha T^{-3/2}\sum_{t=1}^{T}(t-1)\varepsilon_{i,t} + O_p(T^{-1}).
\end{aligned}
$$

We also have

$$
T^{-1/2}\sum_{n=1}^{t}\varepsilon_{i,n} \to_w \sigma_0 W_i(r)
$$

as $T \to \infty$, where $\to_w$ signifies weak convergence and $W_i(r)$ is a standard Brownian motion. It follows that

$$
\begin{aligned}
N_T^{-1}\hat{\delta} &= \begin{bmatrix} \varepsilon_{i,1} \\ (T^{-1/2}\sum_{t=1}^{T}\varepsilon_{i,t} - c_0\alpha T^{-3/2}\sum_{t=1}^{T}(t-1)\varepsilon_{i,t})/h_1(c) \end{bmatrix} + O_p(T^{-1}) \\
&\to_w \begin{bmatrix} \varepsilon_{i,1} \\ \sigma_0(W_i(1) - c_0\alpha\int_{r=0}^{1} r dW_i(r))/h_1(c) \end{bmatrix} \\
&= \begin{bmatrix} \varepsilon_{i,1} \\ \sigma_0\nabla_{0,i}(c) \end{bmatrix}
\end{aligned}
$$

with an obvious definition of $\nabla_{0,i}(c)$. Moreover,

$$
\begin{aligned}
T^{-1/2}N_T\sum_{n=1}^{t}\rho_0^{t-n}d_n &= T^{-1/2}N_T\rho_0^{t-1}d_1 + T^{-1/2}N_T\sum_{n=2}^{t}\rho_0^{t-n}d_n \\
&= T^{-1/2}N_T\sum_{n=2}^{t}\rho_0^{t-n}d_n + O(T^{-1/2}) \\
&= \begin{bmatrix} -(\rho_0 - 1)T^{-1/2}\sum_{n=2}^{t}\rho_0^{t-n} \\ T^{-1}\sum_{n=2}^{t}\rho_0^{t-n}[n - \rho_0(n-1)] \end{bmatrix} + O(T^{-1/2}) \\
&= \begin{bmatrix} -c_0\alpha T^{-3/2}\sum_{n=1}^{t}\rho_0^{t-n} \\ T^{-1}\sum_{n=2}^{t}\rho_0^{t-n}[n - [1 + c_0\alpha T^{-1} + O(T^{-2})](n-1)] \end{bmatrix} + O(T^{-1/2}) \\
&= \begin{bmatrix} -c_0\alpha T^{-3/2}\sum_{n=2}^{t}\rho_0^{t-n} \\ T^{-1}\sum_{n=2}^{t}\rho_0^{t-n} - c_0\alpha T^{-2}\sum_{n=2}^{t}\rho_0^{t-n}n \end{bmatrix} + O(T^{-1/2}) \\
&= \begin{bmatrix} 0 \\ \delta_0(r) \end{bmatrix} + O(T^{-1/2}), \qquad \text{(E.119)}
\end{aligned}
$$

where

$$\delta_0(r) = \int_{u=0}^{r} \phi_0(r-u)du - c_0\alpha \int_{u=0}^{r} u\phi_0(r-u)du.$$

Finally,

$$\begin{aligned}
T^{-1}\text{tr}\, A &= T^{-1}\text{tr}\,(M_{\Gamma_0^{-1}D} L_0 M_{\Gamma_0^{-1}D}) = T^{-1}\text{tr}\,(L_0 M_{\Gamma_0^{-1}D}) \\
&= T^{-1}\text{tr}\, L_0 - T^{-1}\text{tr}\,(L_0\Gamma_0^{-1} D Q^{-1} D' \Gamma_0^{-1\prime}) \\
&= -\text{tr}\,[T^{-1} N_T D' \Gamma_0^{-1\prime} L_0 \Gamma_0^{-1} D N_T (N_T Q N_T)^{-1}] \\
&= -\text{tr} \begin{bmatrix} 0 & 0 \\ 0 & h_2(c)/h_1(c) \end{bmatrix} + O(T^{-1/2}) = -h_2(c)/h_1(c) + O(T^{-1/2})
\end{aligned}$$

Hence, we obtain

$$\xi_{i,N,T} = T^{-1}\varepsilon_i' A \varepsilon_i - \sigma_0^2 T^{-1}\text{tr} A = T^{-1}\varepsilon_i' A\varepsilon_i + h_2(c)/h_1(c) + O(T^{-1/2}). \tag{E.120}$$

Observe that using the structure of L_0, we get

$$\begin{aligned}
T^{-1}\varepsilon_i' A \varepsilon_i &= T^{-1}\left(M_{\Gamma_0^{-1}D}\varepsilon_i\right)' L_0 \left(M_{\Gamma_0^{-1}D}\varepsilon_i\right) = T^{-1}\varepsilon_i^{*\prime} L_0 \varepsilon_i^* \\
&= T^{-1}\sum_{t=2}^{T}\sum_{n=1}^{t-1} \rho_0^{t-n-1}\varepsilon_{i,n}^*\varepsilon_{i,t}^* \\
&= T^{-1}\sum_{t=2}^{T}\sum_{n=1}^{t-1} \rho_0^{t-n-1}\varepsilon_{i,n}\varepsilon_{i,t} - \sum_{t=2}^{T}(N_T^{-1}\hat{\delta})'\left(N_T T^{-1/2}\sum_{n=1}^{t-1}\rho_0^{t-n-1}d_n\right) T^{-1/2}\varepsilon_{i,t} \\
&\quad - \sum_{t=2}^{T}\left(T^{-1/2}\sum_{n=1}^{t-1}\rho_0^{t-n-1}\varepsilon_{i,n}\right)(N_T^{-1}\hat{\delta})' N_T T^{-1/2} d_t \\
&\quad + \sum_{t=2}^{T}(N_T^{-1}\hat{\delta})'\left(N_T T^{-1/2}\sum_{n=1}^{t-1}\rho_0^{t-n-1}d_n\right)(N_T^{-1}\hat{\delta})' N_T T^{-1/2} d_t \\
&= \text{i} - \text{ii} - \text{iii} + \text{iv}.
\end{aligned}$$

with obvious definitions of i $-$ iv. Clearly,

$$\text{i} \to_w \sigma_0^2 \int_0^1 J_i(r) dW_i(r), \tag{E.121}$$

where

$$J_i(r) = \int_0^r \phi_0(r-u) dW_i(u)$$

is a standard Ornstein-Uhlenbeck process. To proceed, given the results above,

$$\text{ii} \to_w \sigma_0^2 \int_0^1 \nabla_{0,i}(c)\delta_0(r) dW_i(r),$$

where, again, $\nabla_{0,i}(c) = \left[W_i(1) - c_0\alpha \int_0^1 udW_i(u)\right] / h_1(c)$. Such integral exists due to the same arguments as in (14.3.30) in Davidson (2000), where the integrand includes de-meaned Brownian motion. For iii, we need some more work. Note that similarly to (E.119) we obtain

$$\begin{aligned} N_T T^{-1/2} \sum_{n=1}^{t} d_n &= N_T T^{-1/2} d_1 + N_T T^{-1/2} \sum_{n=2}^{t} d_n = N_T T^{-1/2} \sum_{n=2}^{t} d_n + O(T^{-1/2}) \\ &= \begin{bmatrix} -(\rho_0 - 1)T^{-1/2}(t-1) \\ T^{-1} \sum_{n=2}^{t} [n - \rho_0(n-1)] \end{bmatrix} + O(T^{-1/2}), \end{aligned} \tag{E.122}$$

where

$$(\rho_0 - 1)T^{-1/2}(t-1) = T^{-3/2} c_0 \alpha t + O(T^{-3/2}) = O(T^{-1/2})$$

because $\sup_{1 \le t \le T} \sup_{(t-1)T^{-1} \le r \le tT^{-1}} |(tT^{-1})^k - r^k| = O(T^{-1})$ for all $k < \infty$. Also

$$\begin{aligned} T^{-1} \sum_{n=2}^{t} [n - \rho_0(n-1)] &= T^{-1} \sum_{n=2}^{t} [n - [1 + c_0\alpha T^{-1} + O(T^{-2})](n-1)] \\ &= T^{-1}(t-1) - c_0\alpha T^{-2} \sum_{n=2}^{t} n + O(T^{-1/2}) \\ &= r - c_0\alpha \int_0^r udu + O(T^{-1/2}) = g_0(r) + O(T^{-1/2}). \end{aligned}$$

Combining the results, we obtain

$$\begin{aligned} \text{iii} &= \sum_{t=2}^{T} \left(T^{-1/2} \sum_{n=1}^{t-1} \rho_0^{t-n-1} \varepsilon_{i,n} \right) (N_T^{-1}\hat{\delta})' N_T T^{-1/2} d_t \\ &\to_w \sigma_0^2 \int_0^1 J_i(r) \nabla_{0,i}(c) dg_0(r) \\ &= \sigma_0^2 \int_0^1 J_i(r) \nabla_{0,i}(c) dr - \sigma_0^2 c_0 \alpha \int_0^1 J_i(r) \nabla_{0,i}(c) r dr \end{aligned} \tag{E.123}$$

using $dg_0(r) = (1 - c_0\alpha r)dr$ in Steltjes integral form. This implies that

$$\begin{aligned} \text{iv} &= \sum_{t=2}^{T} (N_T^{-1}\hat{\delta})' \left(N_T T^{-1/2} \sum_{n=1}^{t-1} \rho_0^{t-n-1} d_n \right) (N_T^{-1}\hat{\delta})' N_T T^{-1/2} d_t \\ &\to_w \sigma_0^2 \int_0^1 \nabla_{0,i}(c)^2 \delta(r) dg_0(r) \\ &= \sigma_0^2 \int_0^1 \nabla_{0,i}(c)^2 \delta(r) dr - \sigma_0^2 c_0 \alpha \int_0^1 \nabla_{0,i}(c)^2 \delta(r) r dr. \end{aligned} \tag{E.124}$$

Overall

$$\begin{aligned} \xi_{i,N,T} &= T^{-1} \varepsilon_i' A \varepsilon_i - \sigma_0^2 T^{-1} \text{tr} A \\ &\to_w \sigma_0^2 \int_0^1 [J_i(r) - \nabla_{0,i}(c)\delta_0(r)] \, dW_i(r) + \sigma_0^2 c_0 \alpha \int_0^1 \left[J_i(r) \nabla_{0,i}(c) - \nabla_{0,i}(c)^2 \delta(r) \right] r dr \\ &\quad + \sigma_0^2 \int_0^1 \left[\nabla_{0,i}(c)^2 \delta(r) - J_i(r) \nabla_{0,i}(c) \right] dr + h_2(c)/h_1(c) \end{aligned} \tag{E.125}$$

This establishes condition (i). As for (ii), we have already shown that $E(\xi_{i,N,T}^2) = \sigma_0^4\omega_0^2 + O(T^{-1/2})$, and it is not difficult to verify that $E(\xi_i^2) = \sigma_0^4\omega_0^2$. This establishes the uniform integrability of $\xi_{i,N,T}^2$, and therefore the Lindeberg condition is satisfied. We can therefore show that

$$\sqrt{N}T^{-1}R_1 \to_d N(0, \sigma_0^4\omega_0^2) \tag{E.126}$$

as N, $T \to \infty$. Next, consider R_2. Recall that,

$$\begin{aligned} \hat{S}_\lambda &= \sigma_0^2(\Gamma_0^{-1}D)^+(\sigma_0^{-2}G - I_T)(\Gamma_0^{-1}D)^{+\prime} \\ &= (D'\Gamma_0^{-1\prime}\Gamma_0^{-1}D)^{-1}D'\Gamma_0^{-1\prime}G\Gamma_0^{-1}D(D'\Gamma_0^{-1\prime}\Gamma_0^{-1}D)^{-1} - \sigma_0^2(D'\Gamma_0^{-1\prime}\Gamma_0^{-1}D)^{-1} \\ &= Q^{-1}D'\Gamma_0^{-1\prime}G\Gamma_0^{-1}DQ^{-1} - \sigma_0^2Q^{-1} \\ &= Q^{-1}D'\Gamma_0^{-1\prime}S_u\Gamma_0^{-1}DQ^{-1} - \sigma_0^2Q^{-1}. \end{aligned} \tag{E.127}$$

By using this and $(A + CBC')^{-1} = A^{-1} - A^{-1}C(B^{-1} + C'A^{-1}C)^{-1}C'A^{-1}$,

$$\begin{aligned} (\sigma_0^2\hat{S}_\lambda^{-1} + Q)^{-1} &= Q^{-1} - Q^{-1}(\sigma^{-2}\hat{S}_\lambda + Q^{-1})^{-1}Q^{-1} \\ &= Q^{-1} - \sigma_0^2(D'\Gamma_0^{-1\prime}S_u\Gamma_0^{-1}D)^{-1}. \end{aligned} \tag{E.128}$$

By using this and $M_{\Gamma_0^{-1}D}\Gamma_0^{-1}D = 0_{T\times 2}$, R_2 can be written as

$$\begin{aligned} R_2 &= \operatorname{tr}[\sigma_0^2(D'\Gamma_0^{-1\prime}S_u\Gamma_0^{-1}D)^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}S_u\Gamma_0^{-1}D] \\ &= \operatorname{tr}([Q^{-1} - (Q^{-1} - \sigma_0^2(D'\Gamma_0^{-1\prime}S_u\Gamma_0^{-1}D)^{-1})]D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}S_u\Gamma_0^{-1}D) \\ &= \operatorname{tr}[HD'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}S_u\Gamma_0^{-1}D] \\ &= \operatorname{tr}[HD'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_u - \sigma_0^2I_T)\Gamma_0^{-1}D], \end{aligned} \tag{E.129}$$

where

$$H = Q^{-1} - (\sigma_0^2\hat{S}_\lambda^{-1} + Q)^{-1}.$$

Also,

$$\begin{aligned} & M_{\Gamma_0^{-1}D}(S_u - \sigma_0^2I_T)\Gamma_0^{-1}D \\ &= M_{\Gamma_0^{-1}D}\Gamma_0^{-1}DS_\lambda D'\Gamma_0^{-1\prime}\Gamma_0^{-1}D + \frac{1}{N}\sum_{i=1}^N M_{\Gamma_0^{-1}D}\Gamma_0^{-1}D\lambda_i\varepsilon_i'\Gamma_0^{-1}D \\ &+ \frac{1}{N}\sum_{i=1}^N M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'D'\Gamma_0^{-1\prime}\Gamma_0^{-1}D + \frac{1}{N}\sum_{i=1}^N M_{\Gamma_0^{-1}D}(\varepsilon_i\varepsilon_i' - \sigma_0^2I_T)\Gamma_0^{-1}D \\ &= \frac{1}{N}\sum_{i=1}^N M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'Q + M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2I_T)\Gamma_0^{-1}D, \end{aligned} \tag{E.130}$$

leading to the following expression for R_2:

$$\begin{aligned} R_2 &= \text{tr}\left(QH\frac{1}{N}\sum_{i=1}^{N} D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'\right) \\ &+ \text{tr}\,[HD'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon-\sigma_0^2 I_T)\Gamma_0^{-1}D] \\ &= R_{21}+R_{22}, \end{aligned} \tag{E.131}$$

with obvious definitions of R_{21} and R_{22}. Consider R_{21}. We begin by noting how

$$\begin{aligned} QHN_T^{-1} &= Q[Q^{-1}-(\sigma_0^2\hat{S}_\lambda^{-1}+Q)^{-1}]N_T^{-1} = [I_2-(\sigma_0^2\hat{S}_\lambda^{-1}Q^{-1}+I_2)^{-1}]N_T^{-1} \\ &= \sigma_0^2\hat{S}_\lambda^{-1}(\sigma_0^2 Q^{-1}\hat{S}_\lambda^{-1}+I_2)^{-1}Q^{-1}N_T^{-1} = \sigma_0^2(\sigma_0^2 Q^{-1}+\hat{S}_\lambda)^{-1}(N_TQ)^{-1}, \end{aligned} \tag{E.132}$$

where we have made use of the fact that $(I_n+AB)^{-1} = I_n - A(I_n+BA)^{-1}B$ for any $n\times n$ matrices A and B.

Now we need to show that $\hat{S}_\lambda$ is consistent for S_λ. Note that expanding S_y for $\rho=\rho_0$ we obtain

$$\begin{aligned} S_y &= \Gamma_0 S_u\Gamma_0' = \Gamma_0\frac{1}{N}\sum_{i=1}^{N}(\Gamma_0^{-1}D\lambda_i+\varepsilon_i)(\Gamma_0^{-1}D\lambda_i+\varepsilon_i)'\Gamma_0' \\ &= \sigma_0^2\Gamma_0\Gamma_0' + DS_\lambda D' + \frac{1}{N}\sum_{i=1}^{N}D\lambda_i\varepsilon_i'\Gamma_0' + \Gamma_0\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i'D' + \Gamma_0(S_\varepsilon-\sigma_0^2 I_T)\Gamma_0'. \end{aligned} \tag{E.133}$$

This implies

$$\begin{aligned} & D'\Gamma^{-1\prime}G\Gamma^{-1}D \\ &= D'\Gamma^{-1\prime}\Gamma^{-1}S_y\Gamma^{-1\prime}\Gamma^{-1}D \\ &= \sigma_0^2 D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D + D'\Gamma^{-1\prime}\Gamma^{-1}DS_\lambda D'\Gamma^{-1\prime}\Gamma^{-1}D \\ &+ D'\Gamma^{-1\prime}\Gamma^{-1}\frac{1}{N}\sum_{i=1}^{N}D\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D + D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i'D'\Gamma^{-1\prime}\Gamma^{-1}D \\ &+ D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon-\sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D \\ &= \sigma_0^2 D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D + QS_\lambda Q + Q\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D \\ &+ D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i'Q + D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon-\sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D. \end{aligned} \tag{E.134}$$

Now, a generic minimizer with respect to S_λ, with $\sigma^2=\sigma_0^2$ and inserting the true model, is given by

$$
\begin{aligned}
\hat{S}_\lambda &= Q^{-1}D'\Gamma^{-1\prime}G\Gamma^{-1}DQ^{-1} - \sigma^2 Q^{-1} \\
&= S_\lambda + \frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1} + Q^{-1}D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i' \\
&+ Q^{-1}D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DQ^{-1} \\
&+ \sigma_0^2 Q^{-1}(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D - Q)Q^{-1} \\
&= S_\lambda + \frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T(N_TQN_T)^{-1}N_T \\
&+ N_T(N_TQN_T)^{-1}N_TD'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i' \\
&+ N_T(N_TQN_T)^{-1}N_TD'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T(N_TQN_T)^{-1}N_T \\
&+ \sigma_0^2 N_T(N_TQN_T)^{-1}N_T(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D - Q)N_T(N_TQN_T)^{-1}N_T. \quad \text{(E.135)}
\end{aligned}
$$

Consider the second term on the right. By using the fact that $\Gamma^{-1}\Gamma_0 = [\Gamma_0^{-1} + (\rho_0 - \rho)J]\Gamma_0 = I_T + (\rho_0 - \rho)L_0$, we have

$$
\begin{aligned}
&\left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T\right\| \\
&= \left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'[I_T + (\rho_0 - \rho)L_0]'\Gamma^{-1}DN_T\right\| \\
&\le \left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T\right\| + |\rho_0 - \rho|\left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'L_0'\Gamma^{-1}DN_T\right\| \\
&\le N^{-1/2}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T\right\| \\
&+ N^{-1/2}T|\rho_0 - \rho|\left\|\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\lambda_i\varepsilon_i'L_0'\Gamma^{-1}DN_T\right\|. \quad \text{(E.136)}
\end{aligned}
$$

Consider $N^{-1/2}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T$. By using $E(\varepsilon_i\varepsilon_i') = \sigma_0^2 I_T$, the cross-section independence of ε_i, the fact that $\varepsilon_i'\Gamma^{-1}DD'\Gamma^{-1\prime}\varepsilon_j$ is just a scalar, and $\|N_T Q N_T\| = O(1)$, we get

$$
\begin{aligned}
& E\left(\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T\right\|^2\right) \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E[\operatorname{tr}(N_T D'\Gamma^{-1\prime}\varepsilon_i\lambda_i'\lambda_j\varepsilon_j'\Gamma^{-1}DN_T)] \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}E(\varepsilon_j'\Gamma^{-1}DN_T N_T D'\Gamma^{-1\prime}\varepsilon_i)\operatorname{tr}(\lambda_j\lambda_i') \\
&= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\operatorname{tr}[E(\varepsilon_i\varepsilon_j')\Gamma^{-1}DN_T N_T D'\Gamma^{-1\prime}]\operatorname{tr}(\lambda_j\lambda_i') \\
&= \frac{1}{N}\sum_{i=1}^{N}\operatorname{tr}[E(\varepsilon_i\varepsilon_i')\Gamma^{-1}DN_T N_T D'\Gamma^{-1\prime}]\operatorname{tr}(\lambda_i\lambda_i') \\
&= \sigma_0^2\operatorname{tr}(N_T D'\Gamma^{-1\prime}\Gamma^{-1}DN_T)\operatorname{tr}(S_\lambda) \\
&= \sigma_0^2\operatorname{tr}(N_T Q N_T)\operatorname{tr}(S_\lambda) = O(1),
\end{aligned}
$$

and, by repeated use of the same argument,

$$
E\left(\left\|\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\lambda_i\varepsilon_i'L_0'\Gamma^{-1}DN_T\right\|^2\right) = \sigma_0^2\operatorname{tr}(T^{-2}N_T D'\Gamma^{-1\prime}L_0L_0'\Gamma^{-1}DN_T)\operatorname{tr}(S_\lambda),
$$

which is $O(1)$ for $\operatorname{tr}(T^{-2}N_T D'\Gamma^{-1\prime}L_0L_0'\Gamma^{-1}DN_T)$ is (see Lemma E.1). It follows that since $\|N^{-1/2}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T\|$ and $\|N^{-1/2}T^{-1}\sum_{i=1}^{N}\lambda_i\varepsilon_i'L_0'\Gamma^{-1}DN_T\|$ are both $O_p(1)$, and $T(\rho_0 - \rho) = (c_0 - c)\alpha + O(T^{-1})$, we have

$$
\begin{aligned}
& \left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T\right\| \\
&\leq N^{-1/2}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma^{-1}DN_T\right\| + N^{-1/2}T|\rho_0-\rho|\left\|\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\lambda_i\varepsilon_i'L_0'\Gamma^{-1}DN_T\right\| \\
&= O_p(N^{-1/2}). \qquad \text{(E.137)}
\end{aligned}
$$

We also know that $\|(N_T Q N_T)^{-1}\| = O(1)$ and $\|N_T\| = 1 + T^{-1/2}$, and so

$$
\begin{aligned}
& \left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T(N_T Q N_T)^{-1}N_T\right\| \\
&\leq N^{-1/2}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T\right\|\|(N_T Q N_T)^{-1}\|\|N_T\| \\
&= O_p(N^{-1/2}). \qquad \text{(E.138)}
\end{aligned}
$$

The order of the third term on the right of (E.135) is the same, because it is just the transpose of the second term.

Let us now consider the fourth term on the right-hand side of (E.135). Note how

$$\begin{aligned} E(\|A(S_\varepsilon - \sigma_0^2 I_T)A'\|^2) &= E(\text{tr}[A(S_\varepsilon - \sigma_0^2 I_T)A'A(S_\varepsilon - \sigma_0^2 I_T)A']) \\ &= E(\text{tr}[AS_\varepsilon A'AS_\varepsilon A' - 2\sigma_0^2 AS_\varepsilon A'AA' + \sigma_0^4 AA'AA']) \\ &= \text{tr}[AE(S_\varepsilon A'AS_\varepsilon)A' - 2\sigma_0^2 AE(S_\varepsilon)A'AA' + \sigma_0^4 AA'AA'] \\ &= \text{tr}[E(AS_\varepsilon A'AS_\varepsilon A') - \sigma_0^4 AA'AA'] \end{aligned}$$

for any deterministic matrix A. Since $\sum_{i=2}^{N} = \sum_{i=1}^{N-1} = N(N-1)/2$ and $\text{tr}(A\varepsilon_i\varepsilon_i'A'A\varepsilon_i\varepsilon_i'A') = (\varepsilon_i'A'A\varepsilon_i)^2$, $\text{tr}[E(AS_\varepsilon A'AS_\varepsilon A')]$ can be written

$$\begin{aligned} &\text{tr}[E(AS_\varepsilon A'AS_\varepsilon A')] \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\text{tr}[E(A\varepsilon_i\varepsilon_i'A'A\varepsilon_i\varepsilon_i'A')] + \frac{2}{N^2}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\text{tr}[AE(\varepsilon_i\varepsilon_i')A'AE(\varepsilon_j\varepsilon_j')A'] \\ &= \frac{1}{N^2}\sum_{i=1}^{N}E[(\varepsilon_i'A'A\varepsilon_i)^2] + N^{-1}(N-1)\sigma_0^4\text{tr}(AA'AA'), \end{aligned}$$

from which it follows that

$$\begin{aligned} E(\|A(S_\varepsilon - \sigma_0^2 I_T)A'\|^2) &= \frac{1}{N^2}\sum_{i=1}^{N}E[(\varepsilon_i'A'A\varepsilon_i)^2] + N^{-1}(N-1)\sigma_0^4\text{tr}(AA'AA') \\ &\quad - \sigma_0^4\text{tr}(AA'AA') \\ &= \frac{1}{N^2}\sum_{i=1}^{N}E[(\varepsilon_i'A'A\varepsilon_i)^2] - N^{-1}\sigma_0^4\text{tr}(AA'AA'). \end{aligned} \tag{E.139}$$

Now set $A = N_T D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0$. We have shown that $\|N_T D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T\| = O(1)$. Therefore, $\text{tr}(AA'AA') = \|AA'\|^2 = O(1)$. A tedious yet straightforward calculation reveals that $N^{-1}\sum_{i=1}^{N}E[(\varepsilon_i'A'A\varepsilon_i)^2] = O(1)$ (see (E.109) for a similar calculation). It follows that

$$E(\|N_T D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T\|^2) = O(N^{-1}),$$

and so

$$\|N_T D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T\| = O_p(N^{-1/2}). \tag{E.140}$$

It remains to consider the last term on the right of (E.135). According to Lemma E.1 and using $\Gamma^{-1}\Gamma_0 = [\Gamma_0^{-1} + (\rho_0 - \rho)J]\Gamma_0 = I_T + (\rho_0 - \rho)L_0$, we obtain

$$
\begin{aligned}
& N_T D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T - N_T Q N_T \\
= \;& N_T D'\Gamma^{-1\prime}(I_T + (\rho_0 - \rho)(L_0 + L_0') + (\rho_0 - \rho)^2 L_0 L_0')\Gamma^{-1}DN_T - N_T Q N_T \\
= \;& (\rho_0 - \rho)N_T D'\Gamma^{-1\prime}(L_0 + L_0')DN_T + (\rho_0 - \rho)^2 N_T D'\Gamma^{-1\prime}L_0 L_0'\Gamma^{-1}DN_T \\
= \;& (c_0 - c)\alpha T^{-1}N_T D'\Gamma^{-1\prime}(L_0 + L_0')\Gamma^{-1}DN_T + (c_0 - c)^2\alpha^2 T^{-2}N_T D'\Gamma^{-1\prime}L_0 L_0'\Gamma^{-1}DN_T \\
+ \;& O(T^{-1}) \\
= \;& \begin{bmatrix} 0 & 0 \\ 0 & 2(c_0 - c)\alpha h_2(c) + (c_0 - c)^2\alpha^2 h_3(c) \end{bmatrix} + O(T^{-1/2}). \qquad \text{(E.141)}
\end{aligned}
$$

Hence, since $\|(N_T Q N_T)^{-1} - \overline{Q}^{-1}\| = O(T^{-1/2})$ with $\overline{Q}^{-1} = \text{diag}[1, 1/h_1(c)]$,

$$
\begin{aligned}
& (N_T Q N_T)^{-1}N_T(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D - Q)N_T(N_T Q N_T)^{-1} \\
= \;& \begin{bmatrix} 0 & 0 \\ 0 & 2(c_0 - c)\alpha h_2(c)/h_1(c)^2 + (c_0 - c)^2\alpha^2 h_3(c)/h_1(c)^2 \end{bmatrix} + O(T^{-1/2}), \qquad \text{(E.142)}
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
& \|N_T(N_T Q N_T)^{-1}N_T(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D - Q)N_T(N_T Q N_T)^{-1}N_T\| \\
= \;& O(T^{-1/2}). \qquad \text{(E.143)}
\end{aligned}
$$

Putting everything together, (E.135) reduces to

$$
\begin{aligned}
\hat{S}_\lambda \;=\; & S_\lambda + \frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T(N_T Q N_T)^{-1}N_T \\
+ \;& N_T(N_T Q N_T)^{-1}N_T D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i' \\
+ \;& N_T(N_T Q N_T)^{-1}N_T D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}DN_T(N_T Q N_T)^{-1}N_T \\
+ \;& \sigma_0^2 N_T(N_T Q N_T)^{-1}N_T(D'\Gamma^{-1\prime}\Gamma^{-1}\Gamma_0\Gamma_0'\Gamma^{-1\prime}\Gamma^{-1}D - Q)N_T(N_T Q N_T)^{-1}N_T \\
= \;& S_\lambda + O_p(N^{-1/2}) + O_p(T^{-1/2}), \qquad \text{(E.144)}
\end{aligned}
$$

which holds uniformly in c.

Now, let us go back to QHN_T^{-1}. Consider $(Q)^{-1}$ and $(QN_T)^{-1}$. Let $Q_{mn} = [Q]_{mn}$ be the element of Q that sits in row n and column m. Since $Q_{12} = Q_{21}$, we have

$$
Q^{-1} = \frac{1}{Q_{22}Q_{11} - Q_{12}^2}\begin{bmatrix} Q_{22} & -Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix} = \frac{T^{-1}Q_{22}}{T^{-1}(Q_{22}Q_{11} - Q_{12}^2)}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + O(T^{-1/2}),
$$

where $T^{-1}Q_{22} = h_1(c) + O(T^{-1/2})$, $Q_{11} = 1 + O(T^{-1/2})$ and $T^{-1/2}Q_{12} = O(T^{-1/2})$ (see Proof of Lemma E.1). Hence, letting

$$
\overline{Q}_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
$$

we can show that

$$Q^{-1} = \overline{Q}_1^{-1} + O(T^{-1/2}). \tag{E.145}$$

We similarly have

$$\begin{aligned}
(QN_T)^{-1} &= \frac{1}{T^{-1/2}(Q_{22}Q_{11} - Q_{12}^2)} \begin{bmatrix} T^{-1/2}Q_{22} & -T^{-1/2}Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix} \\
&= \frac{1}{T^{-1}(Q_{22}Q_{11} - Q_{12}^2)} \begin{bmatrix} T^{-1}Q_{22} & -T^{-1}Q_{12} \\ -T^{-1/2}Q_{12} & T^{-1/2}Q_{11} \end{bmatrix} \\
&= \frac{1}{T^{-1}(Q_{22}Q_{11} - Q_{12}^2)} \begin{bmatrix} T^{-1}Q_{22} & 0 \\ 0 & 0 \end{bmatrix} + O(T^{-1/2}) \\
&= \overline{Q}_1^{-1} + O(T^{-1/2}).
\end{aligned} \tag{E.146}$$

This implies

$$QHN_T^{-1} = \sigma_0^2 \overline{H} + O_p(T^{-1/2}) + O_p(N^{-1/2}), \tag{E.147}$$

where

$$\overline{H} = (\sigma_0^2 \overline{Q}_1^{-1} + S_\lambda)^{-1} \overline{Q}_1^{-1}.$$

A direct calculation reveals that

$$(\sigma_0^2 S_\lambda^{-1} \overline{Q}_1^{-1} + I_2) = \begin{bmatrix} 1 + \sigma_0^2 S_{\lambda 22}/|S_\lambda| & 0 \\ -\sigma_0^2 S_{\lambda 12}/|S_\lambda| & 1 \end{bmatrix}, \tag{E.148}$$

where $|S_\lambda| = S_{\lambda 22} S_{\lambda 11} - S_{\lambda 12}^2$. Hence, letting $d_0 = |\sigma_0^2 S_\lambda^{-1} \overline{Q}_1^{-1} + I_2| = 1 + \sigma_0^2 S_{\lambda 22}/|S_\lambda|$, we obtain

$$(\sigma_0^2 S_\lambda^{-1} \overline{Q}_1^{-1} + I_2)^{-1} = \frac{1}{d_0} \begin{bmatrix} 1 & 0 \\ \sigma_0^2 S_{\lambda 12}/|S_\lambda| & d_0 \end{bmatrix}, \tag{E.149}$$

which in turn implies

$$\begin{aligned}
\overline{H} &= (\sigma_0^2 \overline{Q}_1^{-1} + S_\lambda)^{-1} \overline{Q}_1^{-1} = (\sigma_0^2 S_\lambda^{-1} \overline{Q}_1^{-1} + I_2)^{-1} S_\lambda^{-1} \overline{Q}_1^{-1} \\
&= \frac{1}{d_0 |S_\lambda|} \begin{bmatrix} S_{\lambda 22} & 0 \\ \sigma_0^2 S_{\lambda 12} S_{\lambda 22}/|S_\lambda| - d_0 S_{\lambda 12} & 0 \end{bmatrix} = \frac{1}{d_0 |S_\lambda|} \begin{bmatrix} S_{\lambda 22} & 0 \\ -S_{\lambda 12} & 0 \end{bmatrix}.
\end{aligned} \tag{E.150}$$

The next step in obtaining the variance of R_{21} is to note that

$$\begin{aligned}
& E\left[\operatorname{tr}\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\overline{H}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'\right)^2\right] \\
= \; & E\left[\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\lambda_i'\overline{H}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\varepsilon_i\right)^2\right] \\
= \; & \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_i'\overline{H}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}E(\varepsilon_i\varepsilon_j')M_{\Gamma_0^{-1}D}(L_0+L_0')\Gamma_0^{-1}DN_T\overline{H}'\lambda_j \\
= \; & \sigma_0^2\frac{1}{NT^2}\sum_{i=1}^{N}\lambda_i'\overline{H}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}(L_0+L_0')\Gamma_0^{-1}DN_T\overline{H}'\lambda_i \\
= \; & \sigma_0^2 T^{-2}\operatorname{tr}\left[N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}(L_0+L_0')\Gamma_0^{-1}DN_T\overline{H}'S_\lambda\overline{H}\right] \\
= \; & \sigma_0^2\operatorname{tr}\left[T^{-2}N_T D'\Gamma_0^{-1\prime}(L_0L_0+L_0L_0'+L_0'L_0+L_0'L_0')\Gamma_0^{-1}DN_T\overline{H}'S_\lambda\overline{H}\right] \\
- \; & \sigma_0^2\operatorname{tr}\left[T^{-1}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')\Gamma_0^{-1}DN_T(N_TQN_T)^{-1}\right. \\
\times \; & \left. T^{-1}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')\Gamma_0^{-1}DN_T\overline{H}'S_\lambda\overline{H}\right],
\end{aligned} \tag{E.151}$$

where $\overline{H}'S_\lambda\overline{H}$ has the following simple structure:

$$\overline{H}'S_\lambda\overline{H} = \frac{S_{\lambda 22}}{d_0^2|S_\lambda|}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{E.152}$$

By using this result, $(N_TQN_T)^{-1} = \overline{Q}^{-1} + O(T^{-1/2})$ with $\overline{Q}^{-1} = \operatorname{diag}[1, 1/h_1(c)]$, and Lemma E.1, we get

$$\begin{aligned}
& T^{-1}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')\Gamma_0^{-1}DN_T(N_TQN_T)^{-1}T^{-1}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')\Gamma_0^{-1}DN_T \\
= \; & \frac{4h_2(c)^2}{h_1(c)}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + O(T^{-1/2}),
\end{aligned} \tag{E.153}$$

and

$$\begin{aligned}
& T^{-2}N_T D'\Gamma_0^{-1\prime}(L_0L_0+L_0L_0'+L_0'L_0+L_0'L_0')\Gamma_0^{-1}DN_T \\
= \; & \begin{bmatrix} 0 & 0 \\ 0 & h_3(c)+2h_4(c)+h_5(c) \end{bmatrix} + O(T^{-1/2}).
\end{aligned} \tag{E.154}$$

Due to the structure of these matrices, it is not difficult to see that

$$\begin{aligned}
& E\left[\operatorname{tr}\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\overline{H}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'\right)^2\right] \\
= \; & \sigma_0^2\operatorname{tr}\left[T^{-2}N_T D'\Gamma_0^{-1\prime}(L_0L_0+L_0L_0'+L_0'L_0+L_0'L_0')\Gamma_0^{-1}DN_T\overline{H}'S_\lambda\overline{H}\right] \\
- \; & \sigma_0^2\operatorname{tr}\left[T^{-1}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')\Gamma_0^{-1}DN_T(N_TQN_T)^{-1}\right. \\
\times \; & \left. T^{-1}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')\Gamma_0^{-1}DN_T\overline{H}'S_\lambda\overline{H}\right] \\
= \; & O(T^{-1/2}).
\end{aligned} \tag{E.155}$$

The same results imply

$$
\begin{aligned}
& E\left[\operatorname{tr}\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N} N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'\right)^2\right] \\
= \; & \sigma_0^2 \operatorname{tr}\,[T^{-2}N_T D'\Gamma_0^{-1\prime}(L_0L_0+L_0L_0'+L_0'L_0+L_0'L_0')\Gamma_0^{-1}DN_T S_\lambda] \\
- \; & \sigma_0^2 \operatorname{tr}\,[T^{-1}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')\Gamma_0^{-1}DN_T(N_TQN_T)^{-1}T^{-1}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')\Gamma_0^{-1}DN_T S_\lambda] \\
= \; & \frac{\sigma_0^2 S_{\lambda 22}}{h_1(c)}([h_3(c)+2h_4(c)+h_5(c)]h_1(c)-4h_2(c)^2)+O(T^{-1/2}), \qquad \text{(E.156)}
\end{aligned}
$$

which is $O(1)$. By using this and $\operatorname{tr}\,(A'B)^2 \leq \operatorname{tr}\,(A'A)\operatorname{tr}\,(B'B) = \|A\|^2\|B\|^2$ (see Abadir and Magnus, 2005, Exercise 12.5) and the Cauchy-Schwarz inequality, we have the following:

$$
\begin{aligned}
& E\left[\operatorname{tr}\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}[(QHN_T^{-1})-\sigma_0^2\overline{H}]N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'\right)^2\right] \\
\leq \; & \sqrt{E\left[\|(QHN_T^{-1})-\sigma_0^2\overline{H}\|^4\right]}\sqrt{E\left[\left\|\frac{1}{\sqrt{N}T}\sum_{i=1}^{N} N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'\right\|^4\right]} \\
= \; & O_p(T^{-1})+O_p(N^{-1}),
\end{aligned}
$$

and so, via $(a+b)^2 \leq 2(a^2+b^2)$,

$$
\begin{aligned}
& E[(\sqrt{N}T^{-1}R_{21})^2] \\
= \; & E\left[\operatorname{tr}\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}(QHN_T^{-1})N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'\right)^2\right] \\
\leq \; & 2\sigma_0^4 E\left[\operatorname{tr}\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\overline{H}N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'\right)^2\right] \\
+ \; & 2E\left[\operatorname{tr}\left(\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}[(QHN_T^{-1})-\sigma_0^2\overline{H}]N_T D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\varepsilon_i\lambda_i'\right)^2\right] \\
= \; & O_p(T^{-1/2})+O_p(N^{-1}) \qquad \text{(E.157)}
\end{aligned}
$$

which implies that

$$
\sqrt{N}T^{-1}R_{21} = O_p(N^{-1/2})+O_p(T^{-1/4}). \qquad \text{(E.158)}
$$

Next up is R_{22}, whose variance can be derived using the same steps as for R_1. Specifically, letting $A = \Gamma_0^{-1} D N_T \overline{H}' Q^{-1} D' \Gamma_0^{-1\prime}(L_0 + L_0') M_{\Gamma_0^{-1}D}$, we have

$$
\begin{aligned}
& NT^{-2} E[(\text{tr}\,[N_T \overline{H}' Q^{-1} D' \Gamma_0^{-1\prime}(L_0 + L_0') M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1} D])^2] \\
= \;& NT^{-2} E[(\text{tr}\,[\Gamma_0^{-1} D N_T \overline{H}' Q^{-1} D' \Gamma_0^{-1\prime}(L_0 + L_0') M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)])^2] \\
= \;& NT^{-2} E\left[\left(\text{tr}\left[\frac{1}{N}\sum_{i=1}^{N} A\varepsilon_i \varepsilon_i' - \sigma_0^2 A\right]\right)^2\right] \\
= \;& \frac{1}{NT^2}\sum_{i=1}^{N} E[(\varepsilon_i' A \varepsilon_i - \sigma_0^2 \text{tr} A)^2] + \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j\neq i}^{N} E[(\varepsilon_i' A \varepsilon_i - \sigma_0^2 \text{tr} A)(\varepsilon_j' A \varepsilon_j - \sigma_0^2 \text{tr} A)] \\
= \;& \sigma_0^4 T^{-2}[\text{tr}\,(A'A) + \text{tr}\,(AA)] + O(T^{-1}) \\
= \;& \sigma_0^4 T^{-2} \text{tr}\,[M_{\Gamma_0^{-1}D}(L_0 + L_0')\Gamma_0^{-1} D N_T \overline{H}' Q^{-1} D' \Gamma_0^{-1\prime}\Gamma_0^{-1} D Q^{-1} \overline{H} N_T D' \Gamma_0^{-1\prime}(L_0 + L_0') M_{\Gamma_0^{-1}D}] \\
+ \;& \sigma_0^4 T^{-2} \text{tr}\,[\Gamma_0^{-1} D N_T \overline{H}' Q^{-1} D' \Gamma_0^{-1\prime}(L_0 + L_0') M_{\Gamma_0^{-1}D}\Gamma_0^{-1} D Q^{-1} \overline{H} N_T D' \Gamma_0^{-1\prime}(L_0 + L_0') M_{\Gamma_0^{-1}D}] \\
+ \;& O(T^{-1}) \\
= \;& \sigma_0^4 T^{-2} \text{tr}\,[M_{\Gamma_0^{-1}D}(L_0 + L_0')\Gamma_0^{-1} D N_T \overline{H}' Q^{-1} \overline{H} N_T D' \Gamma_0^{-1\prime}(L_0 + L_0')] + O(T^{-1}) \\
= \;& \sigma_0^4 T^{-2} \text{tr}\,[(L_0 + L_0')\Gamma_0^{-1} D N_T \overline{H}' Q^{-1} \overline{H} N_T D' \Gamma_0^{-1\prime}(L_0 + L_0')] \\
- \;& \sigma_0^4 T^{-2} \text{tr}\,[\Gamma_0^{-1} D Q^{-1} D' \Gamma_0^{-1\prime}(L_0 + L_0')\Gamma_0^{-1} D N_T \overline{H}' Q^{-1} \overline{H} N_T D' \Gamma_0^{-1\prime}(L_0 + L_0')] + O(T^{-1}) \\
= \;& \sigma_0^4 \text{tr}\,[\overline{H}' Q^{-1} \overline{H} T^{-2} N_T D' \Gamma_0^{-1\prime}(L_0 L_0 + L_0 L_0' + L_0' L_0 + L_0' L_0')\Gamma_0^{-1} D N_T] \\
- \;& \sigma_0^4 \text{tr}\,[(N_T Q)^{-1} T^{-1} N_T D' \Gamma_0^{-1\prime}(L_0 + L_0')\Gamma_0^{-1} D N_T \overline{H}' Q^{-1} \overline{H} \\
\times \;& T^{-1} N_T D' \Gamma_0^{-1\prime}(L_0 + L_0')\Gamma_0^{-1} D N_T] + O(T^{-1}),
\end{aligned} \tag{E.159}
$$

where the fourth equality holds because, again,

$$
\begin{aligned}
& E[(\varepsilon_i' A \varepsilon_i - \sigma_0^2 \text{tr}\, A)^2] \\
= \;& E(\varepsilon_i' A \varepsilon_i \varepsilon_i' A \varepsilon_i) - 2\sigma_0^2 E(\varepsilon_i' A \varepsilon_i)\text{tr}\, A + \sigma_0^4 (\text{tr}\, A)^2 \\
= \;& E(\varepsilon_i' A \varepsilon_i \varepsilon_i' A \varepsilon_i) - \sigma_0^4 (\text{tr}\, A)^2 \\
= \;& \sigma_0^4(\kappa_0 - 3)\sum_{t=1}^{T} a_{tt}^2 + \sigma_0^4 \sum_{t=1}^{T}\sum_{n=1}^{T} a_{tt} a_{nn} + \sigma_0^4 \sum_{t=1}^{T}\sum_{n=1}^{T} a_{tn}^2 + \sigma_0^4 \sum_{t=1}^{T}\sum_{n=1}^{T} a_{tn} a_{nt} - \sigma_0^4 \sum_{t=1}^{T}\sum_{n=1}^{T} a_{tt} a_{nn} \\
= \;& \sigma_0^4(\kappa_0 - 3)\sum_{t=1}^{T} a_{tt}^2 + \sigma_0^4 \sum_{t=1}^{T}\sum_{n=1}^{T} a_{tn}^2 + \sigma_0^4 \sum_{t=1}^{T}\sum_{n=1}^{T} a_{tn} a_{nt} \\
= \;& \sigma_0^4(\kappa_0 - 3)\text{tr}\,(A \circ A) + \sigma_0^4 \text{tr}\,(A'A) + \sigma_0^4 \text{tr}\,(AA)
\end{aligned}
$$

with the dominated Hadamard product, and the sixth equality holds because $M_{\Gamma_0^{-1}D}\Gamma_0^{-1} D =$

$0_{T\times 2}$. Also,

$$\begin{aligned}\overline{H}'Q^{-1}\overline{H} &= \overline{H}'\overline{Q}_1^{-1}\overline{H} + O_p(T^{-1/2}) \\ &= \frac{S_{\lambda 22}^2}{d_0^2|S_\lambda|^2}\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + O_p(T^{-1/2}). \end{aligned} \tag{E.160}$$

This result, together with those for $T^{-2}N_T D'\Gamma_0^{-1\prime}(L_0L_0 + L_0L_0' + L_0'L_0 + L_0'L_0')\Gamma_0^{-1}DN_T$ and $T^{-1}N_T D'\Gamma_0^{-1\prime}(L_0 + L_0')\Gamma_0^{-1}DN_T$, implies

$$\begin{aligned}& NT^{-2}E[(\text{tr}\,[N_T\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}D])^2] \\ &= O(T^{-1/2}). \end{aligned} \tag{E.161}$$

Also, similarly to the previous steps, denoting $A = \Gamma_0^{-1}DN_TQ^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}$, we obtain

$$\begin{aligned}& NT^{-2}E[(\text{tr}[Q^{-1}D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}DN_T])^2] \\ &= NT^{-2}E[(\text{tr}[\Gamma_0^{-1}DN_TQ^{-1}D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)])^2] \\ &= NT^{-2}E[(\text{tr}[A(S_\varepsilon - \sigma_0^2 I_T)])^2] \\ &= NT^{-2}E\left[\left(\text{tr}\left[\frac{1}{N}\sum_{i=1}^N A\varepsilon_i\varepsilon_i' - \sigma_0^2 A\right]\right)^2\right] \\ &= \frac{1}{NT^2}\sum_{i=1}^N E[(\varepsilon_i'A\varepsilon_i - \sigma_0^2\text{tr}A)^2] + \frac{1}{NT^2}\sum_{i=1}^N\sum_{j\neq i}^N E[(\varepsilon_i'A\varepsilon_i - \sigma_0^2\text{tr}A)(\varepsilon_j'A\varepsilon_j - \sigma_0^2\text{tr}A)] \\ &= \sigma_0^4 T^{-2}[\text{tr}\,(A'A) + \text{tr}\,(AA)] + O(T^{-1}) \\ &= \sigma_0^4 T^{-2}\text{tr}[M_{\Gamma_0^{-1}D}(L_0+L_0')\Gamma_0^{-1}DQ^{-1}N_TD'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_TQ^{-1}D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}] \\ &+ \sigma_0^4 T^{-2}\text{tr}[\Gamma_0^{-1}DN_TQ^{-1}D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}\Gamma_0^{-1}DN_TQ^{-1}D'\Gamma_0^{-1\prime}(L_0+L_0')M_{\Gamma_0^{-1}D}] \\ &+ O(T^{-1}) \\ &= \sigma_0^4 T^{-2}\text{tr}[M_{\Gamma_0^{-1}D}(L_0+L_0')\Gamma_0^{-1}DQ^{-1}N_TD'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_TQ^{-1}D'\Gamma_0^{-1\prime}(L_0+L_0')] \\ &+ O(T^{-1}) \\ &= \sigma_0^4\text{tr}[(QN_T)^{-1}N_TD'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_T(N_TQ)^{-1}T^{-2}N_TD'\Gamma_0^{-1\prime}(L_0L_0 + L_0L_0' + \\ &+ L_0'L_0 + L_0'L_0')\Gamma_0^{-1}DN_T] \\ &- \sigma_0^4\text{tr}[(N_TQN_T)^{-1}T^{-1}N_TD'\Gamma_0^{-1\prime}(L_0+L_0')\Gamma_0^{-1}DN_T(QN_T)^{-1}N_TD'\Gamma_0^{-1\prime}\Gamma_0^{-1}DN_T \\ &\times (N_TQ)^{-1}T^{-1}N_TD'\Gamma_0^{-1\prime}(L_0+L_0')\Gamma_0^{-1}DN_T] + O(T^{-1}) \\ &= O(T^{-1/2}), \end{aligned} \tag{E.162}$$

which, again, comes from the results for $T^{-2}N_T D'\Gamma_0^{-1\prime}(L_0L_0 + L_0L_0' + L_0'L_0 + L_0'L_0')\Gamma_0^{-1}DN_T$ and $T^{-1}N_T D'\Gamma_0^{-1\prime}(L_0 + L_0')\Gamma_0^{-1}DN_T$, N_TQN_T and $(QN_T)^{-1}$. We can now use the same steps as when evaluating $E[(\sqrt{N}T^{-1}R_{21})^2]$ to show that

$$\begin{aligned}
& E[(\sqrt{N}T^{-1}R_{22})^2] \\
= \; & NT^{-2}E[(\text{tr}\,[HD'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}D])^2] \\
= \; & NT^{-2}E[(\text{tr}\,[N_T(QHN_T^{-1})'Q^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}D])^2] \\
= \; & NT^{-2}E[(\text{tr}[\sigma_0^2 N_T\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}D] \\
+ \; & \text{tr}\left[[(QHN_T^{-1}) - \sigma_0^2\overline{H}]'Q^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}DN_T])^2\right] \\
\leq \; & 2NT^{-2}E[(\text{tr}[\sigma_0^2 N_T\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}D])^2] \\
+ \; & 2NT^{-2}E[(\text{tr}\left[[(QHN_T^{-1}) - \sigma_0^2\overline{H}]'Q^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}DN_T])^2\right] \\
\leq \; & 2NT^{-2}E[(\text{tr}[\sigma_0^2 N_T\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}D])^2] \\
+ \; & 2NT^{-2}E\left[\left\|(QHN_T^{-1}) - \sigma_0^2\overline{H}\right\|^2 \left\|Q^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}DN_T\right\|^2\right] \\
\leq \; & 2NT^{-2}E[(\text{tr}[\sigma_0^2 N_T\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}D])^2] \\
+ \; & 2\sqrt{E\left[\left\|(QHN_T^{-1}) - \sigma_0^2\overline{H}\right\|^4\right]} \\
\times \; & 2\sqrt{E\left[\left\|NT^{-2}Q^{-1}D'\Gamma_0^{-1\prime}(L_0 + L_0')M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)\Gamma_0^{-1}DN_T\right\|^4\right]} \\
= \; & O(T^{-1/2})
\end{aligned} \tag{E.163}$$

Hence, by adding the results for R_{12} and R_{22}, we have that $\sqrt{N}T^{-1}R_2$ is mean zero and with variance

$$E[(\sqrt{N}T^{-1}R_2)^2] = O(T^{-1/2}) + O(N^{-1}), \tag{E.164}$$

implying

$$\sqrt{N}T^{-1}R_2 = O_p(T^{-1/4}) + O_p(N^{-1/2}). \tag{E.165}$$

The results for R_1 and R_2 imply

$$\begin{aligned}
\frac{1}{\sqrt{NT}}\frac{\partial \ell^*}{\partial \rho} \;&=\; \sigma_0^{-2}\sqrt{N}T^{-1}(R_1 + R_2) = \sigma_0^{-2}\sqrt{N}T^{-1}R_1 + O_p(T^{-1/4}) + O_p(N^{-1/2}) \\
&\to_d \; N(0, \omega_0^2)
\end{aligned} \tag{E.166}$$

as N, $T \to \infty$. This establishes part (c).

For (b), since $G = S_u$ and $M_{\Gamma_0^{-1}D} M_{\Gamma_0^{-1}D} = M_{\Gamma_0^{-1}D}$, we have

$$\begin{aligned} \text{tr}\,(GM_{\Gamma_0^{-1}D}) &= \text{tr}\,(S_u M_{\Gamma_0^{-1}D}) = \text{tr}\,(M_{\Gamma_0^{-1}D} S_u M_{\Gamma_0^{-1}D}) = \text{tr}\,(M_{\Gamma_0^{-1}D} S_\varepsilon M_{\Gamma_0^{-1}D}) \\ &= \text{tr}\,(S_\varepsilon M_{\Gamma_0^{-1}D}). \end{aligned} \tag{E.167}$$

Note how $\text{tr}\,(M_{\Gamma_0^{-1}D}) = \text{tr}\, I_T - \text{tr}\,(\Gamma_0^{-1} D Q^{-1} D' \Gamma_0^{-1\prime}) = T - m$, implying

$$E[\text{tr}\,(S_\varepsilon M_{\Gamma_0^{-1}D})] = \text{tr}\,[E(S_\varepsilon) M_{\Gamma_0^{-1}D}] = \sigma_0^2 \text{tr}\,(M_{\Gamma_0^{-1}D}) = \sigma_0^2 (T - m). \tag{E.168}$$

Also,

$$\begin{aligned} \frac{1}{N} \frac{\partial \ell^*}{\partial \sigma^2} &= -\frac{T}{2\sigma_0^2} + \frac{m}{2\sigma_0^2} + \frac{1}{2\sigma_0^4} \text{tr}\,(S_u M_{\Gamma_0^{-1}D}) \\ &= \frac{1}{2\sigma_0^4} [\text{tr}\,(S_\varepsilon M_{\Gamma_0^{-1}D}) - \sigma_0^2 (T - m)] = \frac{1}{2\sigma_0^4} \text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}], \end{aligned} \tag{E.169}$$

or

$$\frac{1}{\sqrt{NT}} \frac{\partial \ell^*}{\partial \sigma^2} = \frac{\sqrt{N}}{2\sigma_0^4 \sqrt{T}} \text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}]. \tag{E.170}$$

This has the same form as in the above analysis of R_1 with $A = M_{\Gamma_0^{-1}D}$. We can therefore use the same steps to show that

$$NT^{-1} E(\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}]^2) = \sigma_0^4 T^{-1} [(\kappa_0 - 3) \text{tr}\,(A \circ A) + \text{tr}\,(A'A) + \text{tr}\,(AA)].$$

Because of the heavier normalization with respect to T, earlier $\text{tr}\,(A \circ A)$ was negligible. This is, however, not the case here. Indeed, from $Q^{-1} = \text{diag}(1, 0) + O(T^{-1/2})$, $(1 - \rho_0) = O(T^{-1})$ and using the notation of Proof of Lemma E.1, we get

$$\begin{aligned} & T^{-1} \text{tr}\,(A \circ A) \\ &= T^{-1} \text{tr}\,([I_T - (1_T - \rho_0 E_1)(1_T - \rho_0 E_1)'] \circ [I_T - (1_T - \rho_0 E_1)(1_T - \rho_0 E_1)']) + O(T^{-1/2}) \\ &= T^{-1}(T-1)[1 - (1 - \rho_0)^2]^2 + O(T^{-1/2}) = 1 + O(T^{-1/2}). \end{aligned} \tag{E.171}$$

Hence, since $T^{-1} \text{tr}\,(M_{\Gamma_0^{-1}D}) = T^{-1}(T - m) = 1 + O(T^{-1})$,

$$\begin{aligned} NT^{-1} E(\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}]^2) &= = \sigma_0^4 T^{-1} [(\kappa_0 - 3) \text{tr}\,(A \circ A) + \text{tr}\,(A'A) + \text{tr}\,(AA)] \\ &= \sigma_0^4 (\kappa_0 - 1) + O(T^{-1}), \end{aligned} \tag{E.172}$$

The arguments used for establishing the asymptotic normality of $(NT)^{-1/2} \partial \ell^* / \partial \sigma^2$ are the same as those used in the analysis of R_1. We can therefore show that

$$\frac{1}{\sqrt{NT}} \frac{\partial \ell^*}{\partial \sigma^2} \to_d N\left(0, \frac{(\kappa_0 - 1)}{4\sigma_0^4}\right) \tag{E.173}$$

as $N, T \to \infty$.

In (c), we show that $\partial \ell^*/\partial \rho$ and $\partial \ell^*/\partial \sigma^2$ are asymptotically independent. We begin by noting how

$$E\left(\frac{1}{NT^{3/2}}\frac{\partial \ell^*}{\partial \sigma^2}\frac{\partial \ell^*}{\partial \rho}\right) = \frac{N}{2\sigma_0^4 T^{3/2}} E(\mathrm{tr}\,[(S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}] R_1) + O(T^{-1/4}) + O(N^{-1/4}). \tag{E.174}$$

Letting $A_1 = M_{\Gamma_0^{-1}D}$ and $A_2 = M_{\Gamma_0^{-1}D} L_0 M_{\Gamma_0^{-1}D}$, the first term on the right becomes

$$\begin{aligned} & NT^{-1} E(\mathrm{tr}\,[(S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}] R_1) \\ &= NT^{-1} E(\mathrm{tr}\,[(S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}] \mathrm{tr}\,[L_0 M_{\Gamma_0^{-1}D} (S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}]) \\ &= \frac{1}{NT} \sum_{i=1}^N \sum_{j=1}^N E[(\varepsilon_i' A_1 \varepsilon_i - \sigma_0^2 \mathrm{tr}\, A_1)(\varepsilon_j' A_2 \varepsilon_j - \sigma_0^2 \mathrm{tr}\, A_2)] \\ &= \frac{1}{NT} \sum_{i=1}^N E[(\varepsilon_i' A_1 \varepsilon_i - \sigma_0^2 \mathrm{tr}\, A_1)(\varepsilon_i' A_2 \varepsilon_i - \sigma_0^2 \mathrm{tr}\, A_2)], \end{aligned} \tag{E.175}$$

where

$$\begin{aligned} & E[(\varepsilon_i' A_1 \varepsilon_i - \sigma_0^2 \mathrm{tr}\, A_1)(\varepsilon_i' A_2 \varepsilon_i - \sigma_0^2 \mathrm{tr}\, A_2)] \\ &= E(\varepsilon_i' A_1 \varepsilon_i \varepsilon_i' A_2 \varepsilon_i) - E(\varepsilon_i' A_1 \varepsilon_i) \sigma_0^2 \mathrm{tr}\, A_2 - \sigma_0^2 \mathrm{tr}\, A_1 E(\varepsilon_i' A_2 \varepsilon_i) + \sigma_0^4 \mathrm{tr}\, A_1 \mathrm{tr}\, A_2 \\ &= E(\varepsilon_i' A_1 \varepsilon_i \varepsilon_i' A_2 \varepsilon_i) - \sigma_0^4 \mathrm{tr}\, A_1 \mathrm{tr}\, A_2. \end{aligned} \tag{E.176}$$

Hence, in analogy with the analysis of R_1,

$$\begin{aligned} & T^{-1} E[(\varepsilon_i' A_1 \varepsilon_i - \sigma_0^2 \mathrm{tr}\, A_1)(\varepsilon_i' A_2 \varepsilon_i - \sigma_0^2 \mathrm{tr}\, A_2)] \\ &= \sigma_0^4 T^{-1} [(\kappa_0 - 3) \mathrm{tr}\,(A_1 \circ A_2) + \mathrm{tr}\,(A_1' A_2) + \mathrm{tr}\,(A_1 A_2)], \end{aligned} \tag{E.177}$$

where $\mathrm{tr}\,(A_1' A_2) = \mathrm{tr}\,(A_1 A_2) = 2\mathrm{tr}\,(M_{\Gamma_0^{-1}D} L_0 M_{\Gamma_0^{-1}D}) = O(T)$. Also, $T^{-1}\mathrm{tr}\,(M_{\Gamma_0^{-1}D} L_0 M_{\Gamma_0^{-1}D})$ dominates $T^{-1}\mathrm{tr}\,(A_1 \circ A_2)$. It follows that

$$\begin{aligned} E\left(\frac{1}{NT^{3/2}}\frac{\partial \ell^*}{\partial \sigma^2}\frac{\partial \ell^*}{\partial \rho}\right) &= \frac{N}{2\sigma_0^4 T^{3/2}} E(\mathrm{tr}\,[(S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}] R_1) \\ &+ O(T^{-1/4}) + O(N^{-1/4}) \\ &= T^{-3/2} \mathrm{tr}\,(M_{\Gamma_0^{-1}D} L_0 M_{\Gamma_0^{-1}D}) + O(T^{-1/4}) + O(N^{-1/4}) \\ &= O(T^{-1/4}) + O(N^{-1/4}). \end{aligned} \tag{E.178}$$

Therefore, $N^{-1/2}T^{-1}\partial \ell^*/\partial \rho$ and $(NT)^{-1/2}\partial \ell^*/\partial \sigma^2$ are asymptotically uncorrelated, and hence independent by (asymptotic) normality. This implies

$$(NT)^{-1/2} N_T \frac{\partial \ell^*}{\partial \theta_2} = \begin{bmatrix} (NT)^{-1/2} \frac{\partial \ell^*}{\partial \sigma^2} \\ N^{-1/2} T^{-1} \frac{\partial \ell^*}{\partial \rho} \end{bmatrix} \to_d N\left(0_{2\times 1}, \begin{bmatrix} (\kappa_0 - 1)/(4\sigma_0^4) & 0 \\ 0 & \omega_0^2 \end{bmatrix}\right) \tag{E.179}$$

as N, $T \to \infty$.

For (d), by using the fact that $G = S_u$ and $M_{\Gamma_0^{-1}D} S_u M_{\Gamma_0^{-1}D} = M_{\Gamma_0^{-1}D} S_\varepsilon M_{\Gamma_0^{-1}D}$, r_1 can be written as

$$\begin{aligned} r_1 &= -\sigma_0^2 \mathrm{tr}\,[(L_0' + L_0) M_{\Gamma_0^{-1}D} L_0 M_{\Gamma_0^{-1}D}] \\ &- \mathrm{tr}\,[M_{\Gamma_0^{-1}D}(G - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}(L_0' + L_0) M_{\Gamma_0^{-1}D}(L_0' + L_0)] \\ &+ \mathrm{tr}\,[M_{\Gamma_0^{-1}D}(G - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D} L_0 (L_0' + 2L_0)] \\ &= -\sigma_0^2 \mathrm{tr}\,[(L_0' + L_0) M_{\Gamma_0^{-1}D} L_0 M_{\Gamma_0^{-1}D}] \\ &- \mathrm{tr}\,[M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D}(L_0' + L_0) M_{\Gamma_0^{-1}D}(L_0' + L_0)] \\ &+ \mathrm{tr}\,[M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T) M_{\Gamma_0^{-1}D} L_0 (L_0' + 2L_0)]. \end{aligned}$$

Now let $A = M_{\Gamma_0^{-1}D} L_0 (L_0' + 2L_0) M_{\Gamma_0^{-1}D} - M_{\Gamma_0^{-1}D}(L_0' + L_0) M_{\Gamma_0^{-1}D}(L_0' + L_0) M_{\Gamma_0^{-1}D}$ and apply the same steps as when evaluating R_1 to show that

$$\begin{aligned} T^{-2} r_1 &= -\sigma_0^2 T^{-2} \mathrm{tr}\,[(L_0' + L_0) M_{\Gamma_0^{-1}D} L_0 M_{\Gamma_0^{-1}D}] + T^{-2} \mathrm{tr}\,[(S_\varepsilon - \sigma_0^2 I_T) A] \\ &= -\sigma_0^2 T^{-2} \mathrm{tr}\,[(L_0' + L_0) M_{\Gamma_0^{-1}D} L_0 M_{\Gamma_0^{-1}D}] + O_p(N^{-1/2}) \\ &= -\sigma_0^2 \omega_0^2 + O_p(T^{-1/2}) + O_p(N^{-1/2}). \end{aligned} \tag{E.180}$$

Next up is r_2, which we write as

$$\begin{aligned} r_2 &= \mathrm{tr}\,[\sigma_0^2 (D'\Gamma_0^{-1\prime} G \Gamma_0^{-1} D)^{-1} (D'\Gamma_0^{-1\prime}(L_0' + L_0) G \Gamma_0^{-1} D + D'\Gamma_0^{-1\prime} G (L_0' + L_0) \Gamma_0^{-1} D) \\ &\times (D'\Gamma_0^{-1\prime} G \Gamma_0^{-1} D)^{-1} D'\Gamma_0^{-1\prime}(L_0' + L_0) M_{\Gamma_0^{-1}D} G \Gamma_0^{-1} D \\ &+ \sigma_0^2 (D'\Gamma_0^{-1\prime} G \Gamma_0^{-1} D)^{-1} (D'\Gamma_0^{-1\prime}(L_0' + L_0)(L_0' + L_0) M_{\Gamma_0^{-1}D} G \Gamma_0^{-1} D \\ &- D'\Gamma_0^{-1\prime}(L_0' + L_0) M_{\Gamma_0^{-1}D}(L_0' + L_0) M_{\Gamma_0^{-1}D} G \Gamma_0^{-1} D \\ &- D'\Gamma_0^{-1\prime}(L_0' + L_0) M_{\Gamma_0^{-1}D} G (L_0' + L_0) \Gamma_0^{-1} D - 2D'\Gamma_0^{-1\prime} L_0' L_0 M_{\Gamma_0^{-1}D} G \Gamma_0^{-1} D))] \\ &= \sigma_0^{-2} \mathrm{tr}\,[H D'\Gamma_0^{-1\prime}(L_0' + L_0) G \Gamma_0^{-1} D H D'\Gamma_0^{-1\prime}(L_0' + L_0) M_{\Gamma_0^{-1}D} G \Gamma_0^{-1} D] \\ &+ \sigma_0^{-2} \mathrm{tr}\,[H D'\Gamma_0^{-1\prime} G (L_0' + L_0) \Gamma_0^{-1} D H D'\Gamma_0^{-1\prime}(L_0' + L_0) M_{\Gamma_0^{-1}D} G \Gamma_0^{-1} D] \\ &+ \mathrm{tr}\,[H D'\Gamma_0^{-1\prime}(L_0' + L_0)(L_0' + L_0) M_{\Gamma_0^{-1}D} G \Gamma_0^{-1} D] \\ &- \mathrm{tr}\,[H D'\Gamma_0^{-1\prime}(L_0' + L_0) M_{\Gamma_0^{-1}D}(L_0' + L_0) M_{\Gamma_0^{-1}D} G \Gamma_0^{-1} D] \\ &- \mathrm{tr}\,[H D'\Gamma_0^{-1\prime}(L_0' + L_0) M_{\Gamma_0^{-1}D} G (L_0' + L_0) \Gamma_0^{-1} D] \\ &- 2\mathrm{tr}\,(H D'\Gamma_0^{-1\prime} L_0' L_0 M_{\Gamma_0^{-1}D} G \Gamma_0^{-1} D). \end{aligned}$$

Further use of $M_{\Gamma_0^{-1}D}\Gamma_0^{-1}D = 0_{T\times 2}$ and $G = S_u$ gives

$$\begin{aligned} r_2 &= 2\sigma_0^{-2}\text{tr}\,[HD'\Gamma_0^{-1\prime}(L_0' + L_0)S_u\Gamma_0^{-1}DHD'\Gamma_0^{-1\prime}(L_0' + L_0)M_{\Gamma_0^{-1}D}(S_u - \sigma_0^2 I_T)\Gamma_0^{-1}D] \\ &+ \text{tr}\,[HD'\Gamma_0^{-1\prime}(L_0' + L_0)(L_0' + L_0)M_{\Gamma_0^{-1}D}(S_u - \sigma_0^2 I_T)\Gamma_0^{-1}D] \\ &- \text{tr}\,[HD'\Gamma_0^{-1\prime}(L_0' + L_0)M_{\Gamma_0^{-1}D}(L_0' + L_0)M_{\Gamma_0^{-1}D}(S_u - \sigma_0^2 I_T)\Gamma_0^{-1}D] \\ &- \text{tr}\,[HD'\Gamma_0^{-1\prime}(L_0' + L_0)M_{\Gamma_0^{-1}D}(S_u - \sigma_0^2 I_T)(L_0' + L_0)\Gamma_0^{-1}D] \\ &- 2\text{tr}\,[HD'\Gamma_0^{-1\prime}L_0'L_0M_{\Gamma_0^{-1}D}(S_u - \sigma_0^2 I_T)\Gamma_0^{-1}D]. \end{aligned} \tag{E.181}$$

The last four terms on the right-hand side have the same form as R_2. We can therefore use the same steps as before to show that their variances are $O(T^{-1/2}) + O(N^{-1/2})$ when scaled by $\sqrt{N}T^{-2}$. It follows that

$$\begin{aligned} T^{-2}r_2 &= 2\sigma_0^{-2}T^{-2}\text{tr}\,[HD'\Gamma_0^{-1\prime}(L_0' + L_0)S_u\Gamma_0^{-1}DHD'\Gamma_0^{-1\prime}(L_0' + L_0)M_{\Gamma_0^{-1}D}(S_u - \sigma_0^2 I_T)\Gamma_0^{-1}D] \\ &+ O_p(N^{-1/2}T^{-1/4}) + O_p(N^{-3/4}) \\ &= 2\sigma_0^{-2}\text{tr}\,[T^{-1}N_T^{-1}HD'\Gamma_0^{-1\prime}(L_0' + L_0)S_u\Gamma_0^{-1}DHN_T^{-1} \\ &\times T^{-1}N_TD'\Gamma_0^{-1\prime}(L_0' + L_0)M_{\Gamma_0^{-1}D}(S_u - \sigma_0^2 I_T)\Gamma_0^{-1}DN_T] + O_p(N^{-1/2}T^{-1/4}) \\ &+ O_p(N^{-3/4}) \\ &= 2\sigma_0^{-2}\text{tr}\,[T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0' + L_0)S_u\Gamma_0^{-1}DQ^{-1}\overline{H} \\ &\times T^{-1}N_TD'\Gamma_0^{-1\prime}(L_0' + L_0)M_{\Gamma_0^{-1}D}(S_u - \sigma_0^2 I_T)\Gamma_0^{-1}DN_T] + O_p(N^{-1/2}) \\ &+ O_p(T^{-1/2}). \end{aligned} \tag{E.182}$$

From

$$S_u = \Gamma_0^{-1}DS_\lambda D'\Gamma_0^{-1\prime} + \frac{1}{N}\sum_{i=1}^N \Gamma_0^{-1}D\lambda_i\varepsilon_i' + \frac{1}{N}\sum_{i=1}^N \varepsilon_i\lambda_i'D'\Gamma_0^{-1\prime} + S_\varepsilon,$$

we get

$$\begin{aligned} &T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0' + L_0)S_u\Gamma_0^{-1}DQ^{-1}\overline{H} \\ &\quad= \sigma_0^2 T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0' + L_0)\Gamma_0^{-1}DS_\lambda\overline{H} \\ &\quad+ \sigma_0^2 T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0' + L_0)\Gamma_0^{-1}D\frac{1}{N}\sum_{i=1}^N \lambda_i\varepsilon_i'\Gamma_0^{-1}DQ^{-1}\overline{H} \\ &\quad+ \sigma_0^2 T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0' + L_0)\frac{1}{N}\sum_{i=1}^N \varepsilon_i\lambda_i'\overline{H} \\ &\quad+ \sigma_0^2 T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0' + L_0)S_\varepsilon\Gamma_0^{-1}DQ^{-1}\overline{H}. \end{aligned} \tag{E.183}$$

Note how $\lambda_i'\overline{H}\overline{H}'\lambda_j'$ is a scalar. By using this and some of the previously obtained results, we can show that

$$
\begin{aligned}
& E\left(\left\|T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0'+L_0)\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i\lambda_i'\overline{H}\right\|^2\right) \\
= \; & \frac{1}{(NT)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\text{tr}\,[\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0'+L_0)E(\varepsilon_i\varepsilon_j')(L_0'+L_0)\Gamma_0^{-1}DQ^{-1}\overline{H}]\lambda_i'\overline{H}\overline{H}'\lambda_j \\
= \; & \sigma_0^2N^{-1}\text{tr}\,[\overline{H}'(N_TQ)^{-1}T^{-2}N_TD'\Gamma_0^{-1\prime}(L_0'L_0'+L_0'L_0+L_0L_0'+L_0L_0)\Gamma_0^{-1}DN_T(QN_T)^{-1}\overline{H}] \\
\times \; & \text{tr}\,(S_\lambda\overline{H}\overline{H}') \\
= \; & \sigma_0^2N^{-1}\text{tr}\,[\overline{H}'\overline{Q}_1^{-1}T^{-2}N_TD'\Gamma_0^{-1\prime}(L_0'L_0'+L_0'L_0+L_0L_0'+L_0L_0)\Gamma_0^{-1}DN_T\overline{Q}_1^{-1}\overline{H}]\text{tr}\,(\overline{H}'S_\lambda\overline{H}) \\
+ \; & O(N^{-1}T^{-1/2}) \\
= \; & O(T^{-1/2}).
\end{aligned}
$$

Similarly, since $\varepsilon_i'\Gamma_0^{-1}DQ^{-1}\overline{H}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}\varepsilon_j$ is a scalar,

$$
\begin{aligned}
& E\left(\left\|T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}D\frac{1}{N}\sum_{i=1}^{N}\lambda_i\varepsilon_i'\Gamma_0^{-1}DQ^{-1}\overline{H}\right\|^2\right) \\
= \; & \frac{1}{(NT)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\text{tr}\,[\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}D\lambda_i\lambda_j'D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DQ^{-1}\overline{H}] \\
\times \; & E(\varepsilon_i'\Gamma_0^{-1}DQ^{-1}\overline{H}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}\varepsilon_j) \\
= \; & \frac{1}{(NT)^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\text{tr}\,[\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}D\lambda_i\lambda_j'D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DQ^{-1}\overline{H}] \\
\times \; & \text{tr}\,[\Gamma_0^{-1}DQ^{-1}\overline{H}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}E(\varepsilon_j\varepsilon_i')] \\
= \; & \sigma_0^2N^{-1}\text{tr}\,[T^{-2}\overline{H}'(N_TQ)^{-1}N_TD'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DS_\lambda \\
\times \; & D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DN_T(QN_T)^{-1}\overline{H}]\text{tr}\,(\overline{H}'Q^{-1}\overline{H}) \\
= \; & \sigma_0^2N^{-1}\text{tr}\,[\overline{H}'\overline{Q}_1^{-1}T^{-2}N_TD'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DS_\lambda D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DN_T\overline{Q}_1^{-1}\overline{H}] \\
\times \; & \text{tr}\,(\overline{H}'\overline{Q}_1^{-1}\overline{H})+O(N^{-1}T^{-1/2}) \\
= \; & O(N^{-1}),
\end{aligned}
$$

where the last equality holds since while we know that $\text{tr}\,(\overline{H}'\overline{Q}_1^{-1}\overline{H})$ is equal to a constant, the other trace converges to one. The proof of this last result is straightforward but tedious, as we are evaluating the whole of $T^{-2}N_TD'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DS_\lambda D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DN_T$.

The same argument can be used to show that

$$
\begin{aligned}
& \|T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DS_\lambda\overline{H}\|^2 \\
= & \ \mathrm{tr}\,[T^{-2}\overline{H}'(N_TQ)^{-1}N_TD'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DS_\lambda\overline{H}\overline{H}'S_\lambda \\
\times & \ D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DN_T(QN_T)^{-1}\overline{H}] \\
= & \ \mathrm{tr}\,[T^{-2}\overline{H}'\overline{Q}_1^{-1}N_TD'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DS_\lambda\overline{H}\overline{H}'S_\lambda \\
\times & \ D'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DN_T\overline{Q}_1^{-1}\overline{H}] + O_p(T^{-1/2}) \\
= & \ O_p(1)
\end{aligned}
$$

and

$$
\begin{aligned}
& \|T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0'+L_0)S_\varepsilon\Gamma_0^{-1}DQ^{-1}\overline{H}\|^2 \\
= & \ \mathrm{tr}\,[\overline{H}'N_T(N_TQN_T)^{-1}T^{-1}N_TD'\Gamma_0^{-1\prime}(L_0'+L_0)S_\varepsilon\Gamma_0^{-1}DN_T(N_TQN_T)^{-1}N_T\overline{H} \\
\times & \ \overline{H}N_T(N_TQN_T)^{-1}N_TD'\Gamma_0^{-1\prime}S_\varepsilon(L_0'+L_0)\Gamma_0^{-1}DN_T(N_TQN_T)^{-1}N_T\overline{H}] \\
= & \ \sigma_0^4\mathrm{tr}\,[\overline{H}'N_T\overline{Q}^{-1}T^{-1}N_TD'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DN_T\overline{Q}^{-1}N_T\overline{H} \\
\times & \ \overline{H}N_T\overline{Q}^{-1}T^{-1}N_TD'\Gamma_0^{-1\prime}(L_0'+L_0)\Gamma_0^{-1}DN_T\overline{Q}^{-1}N_T\overline{H}] + O_p(N^{-1/2}) + O_p(T^{-1/2}) \\
= & \ O_p(1).
\end{aligned}
$$

Hence, by collecting the terms,

$$
\|T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0'+L_0)S_u\Gamma_0^{-1}DQ^{-1}\overline{H}\| = O_p(1). \tag{E.184}
$$

The second matrix product in $T^{-2}r_2$ is $T^{-1}N_TH^{-1}R_2N_T$. This matrix can be shown to be $O_p(N^{-1/2})$, implying that $T^{-2}r_2$ reduces to

$$
\begin{aligned}
T^{-2}r_2 = & \ 2\sigma_0^{-2}\mathrm{tr}\,[T^{-1}\overline{H}'Q^{-1}D'\Gamma_0^{-1\prime}(L_0'+L_0)S_u\Gamma_0^{-1}DQ^{-1}\overline{H} \\
\times & \ T^{-1}N_TD'\Gamma_0^{-1\prime}(L_0'+L_0)M_{\Gamma_0^{-1}D}(S_u-\sigma_0^2I_T)\Gamma_0^{-1}DN_T] + O_p(N^{-1/2}) + O_p(T^{-1/2}) \\
= & \ O_p(N^{-1/2}) + O_p(T^{-1/2}).
\end{aligned} \tag{E.185}
$$

Putting everything together,

$$
\frac{1}{NT^2}\frac{\partial^2\ell^*}{(\partial\rho)^2} \to_p -\omega_0^2, \tag{E.186}
$$

as $N, T \to \infty$. This establishes part (d).

We move on to consider (e). By using the results obtained for R_1,

$$
\begin{aligned}
&T^{-3/2}\text{tr}\,(M_{\Gamma_0^{-1}D}GM_{\Gamma_0^{-1}D}L_0) \\
&= T^{-3/2}\text{tr}\,(M_{\Gamma_0^{-1}D}S_u M_{\Gamma_0^{-1}D}L_0) = T^{-3/2}\text{tr}\,(M_{\Gamma_0^{-1}D}S_\varepsilon M_{\Gamma_0^{-1}D}L_0) \\
&= \sigma_0^2 T^{-3/2}\text{tr}\,(M_{\Gamma_0^{-1}D}L_0) + T^{-3/2}\text{tr}\,[M_{\Gamma_0^{-1}D}(S_\varepsilon - \sigma_0^2 I_T)M_{\Gamma_0^{-1}D}L_0] \\
&= O_p(T^{-1/2}). \qquad \text{(E.187)}
\end{aligned}
$$

Finally, consider (f). By adding and subtracting appropriately, we obtain

$$
\begin{aligned}
\frac{1}{N}\frac{\partial^2 \ell^*}{(\partial\sigma^2)^2} &= \frac{T}{2\sigma_0^4} - \frac{m}{2\sigma_0^4} - \sigma_0^{-6}\text{tr}\,(GM_{\Gamma^{-1}D}) \\
&= -\frac{T}{2\sigma_0^4} + \frac{m}{2\sigma_0^2} + \frac{T}{\sigma_0^4} - \frac{m}{\sigma_0^4} - \sigma_0^{-6}\text{tr}\,(GM_{\Gamma^{-1}D}) \\
&= -\frac{T}{2\sigma_0^4} + \frac{m}{2\sigma_0^4} - \frac{1}{\sigma_0^6}\left(\text{tr}\,(GM_{\Gamma^{-1}D}) - \sigma_0^2(T-m)\right) \\
&= -\frac{T}{2\sigma_0^4} + \frac{m}{2\sigma_0^4} - \frac{1}{\sigma_0^6}\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)M_{\Gamma_0^{-1}D}]. \qquad \text{(E.188)}
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\frac{1}{NT}\frac{\partial^2 \ell^*}{(\partial\sigma^2)^2} &= -\frac{1}{2\sigma_0^4} + \frac{m}{2T\sigma_0^4} - \frac{1}{T\sigma_0^6}\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)M_{\Gamma_0^{-1}D}] \\
&= = -\frac{1}{2\sigma_0^4} - (NT)^{-1/2}\frac{\sqrt{N}}{\sqrt{T}\sigma_0^6}\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)M_{\Gamma_0^{-1}D}] + O(T^{-1}) \\
&= -\frac{1}{2\sigma_0^4} + O(T^{-1}) + O((NT)^{-1/2}), \qquad \text{(E.189)}
\end{aligned}
$$

because

$$
\begin{aligned}
NT^{-1}E(\text{tr}\,[(S_\varepsilon - \sigma_0^2 I_T)M_{\Gamma_0^{-1}D}]^2) &= = \sigma_0^4 T^{-1}[(\kappa_0 - 3)\text{tr}\,(A\circ A) + \text{tr}\,(A'A) + \text{tr}\,(AA)] \\
&= \sigma_0^4(\kappa_0 - 1) + O(T^{-1}), \qquad \text{(E.190)}
\end{aligned}
$$

following steps up to (E.172). ■

Proof of Theorem 2.

According to Lemma E.2,

$$
\text{(a)} \quad N_T \frac{1}{\sqrt{NT}}\frac{\partial \ell^*(\theta_2^0)}{\partial\theta_2} \to_d N\left(0_{2\times 1}, \lim_{N,T\to\infty}\begin{bmatrix} (\kappa_0 - 1)/(4\sigma_0^4) & 0 \\ 0 & s_0^2 \end{bmatrix}\right),
$$

$$
\text{(b)} \quad -N_T \frac{1}{NT}\frac{\partial^2 \ell^*(\theta_2^0)}{\partial\theta_2(\partial\theta_2)'}N_T \to_p \lim_{N,T\to\infty}\begin{bmatrix} 1/(2\sigma_0^4) & 0 \\ 0 & s_0^2 \end{bmatrix},
$$

where $s_0^2 = s^2(\rho_0) = T^{-2}\text{tr}\left[(L(\rho_0)' + L(\rho_0))M_{\Gamma(\rho_0)^{-1}D}L(\rho_0)M_{\Gamma(\rho_0)^{-1}D}\right]$. Similarly to Norkutė and Westerlund (2021), we begin by applying the mean value theorem to the score $\partial\ell^*(\hat{\theta}_2)/\partial\theta_2$ around $\hat{\theta}_2 = \theta_2^0$ (see, for example, Newey et al., 1994, page 2141). This gives

$$0_{2\times 1} = \frac{\partial \ell^*(\hat{\theta}_2)}{\partial\theta_2} = \frac{\partial \ell^*(\theta_2^0)}{\partial\theta_2} + \frac{\partial^2 \ell^*(\overline{\theta}_2)}{\partial\theta_2(\partial\theta_2)'}(\hat{\theta}_2 - \theta_2^0), \tag{E.191}$$

where $\overline{\theta}_2$ lies element-wise between the line segment joining $\hat{\theta}_2$ and θ_2^0. Suppose that the Hessian is invertible. Under this assumption,

$$\begin{aligned}
\sqrt{NT}N_T^{-1}(\hat{\theta}_2 - \theta_2^0) &= \left(-N_T\frac{1}{NT}\frac{\partial^2 \ell^*(\overline{\theta}_2)}{\partial\theta_2(\partial\theta_2)'}N_T\right)^{-1} N_T\frac{1}{\sqrt{NT}}\frac{\partial \ell^*(\theta_2^0)}{\partial\theta_2} \\
&= \left(-N_T\frac{1}{NT}\frac{\partial^2 \ell^*(\theta_2^0)}{\partial\theta_2(\partial\theta_2)'}N_T\right)^{-1} N_T\frac{1}{\sqrt{NT}}\frac{\partial \ell^*(\theta_2^0)}{\partial\theta_2} \\
&+ \left[\left(-N_T\frac{1}{NT}\frac{\partial^2 \ell^*(\overline{\theta}_2)}{\partial\theta_2(\partial\theta_2)'}N_T\right)^{-1} - \left(-N_T\frac{1}{NT}\frac{\partial^2 \ell^*(\theta_2^0)}{\partial\theta_2(\partial\theta_2)'}N_T\right)^{-1}\right] N_T\frac{1}{\sqrt{NT}}\frac{\partial \ell^*(\theta_2^0)}{\partial\theta_2} \\
&= \left(-N_T\frac{1}{NT}\frac{\partial^2 \ell^*(\theta_2^0)}{\partial\theta_2(\partial\theta_2)'}N_T\right)^{-1} N_T\frac{1}{\sqrt{NT}}\frac{\partial \ell^*(\theta_2^0)}{\partial\theta_2} + o_p(1),
\end{aligned} \tag{E.192}$$

where, again, $N_T = \text{diag}(1, T^{-1/2})$. This expansion holds uniformly on Θ_2. Here,

$$\left\| N_T\frac{1}{NT}\frac{\partial^2 \ell^*(\overline{\theta}_2)}{\partial\theta_2(\partial\theta_2)'}N_T - N_T\frac{1}{NT}\frac{\partial^2 \ell^*(\theta_2^0)}{\partial\theta_2(\partial\theta_2)'}N_T \right\| = o_p(1) \tag{E.193}$$

follows from stochastic equi-continuity (see Newey et al., 1994, page 2137, for a definition). This can be verified using consistency of $\hat{\theta}_2$ and analytical expression of the first and second order derivatives of $\ell^*(\theta_2)$ (see the proof of Theorem 3 in Hayakawa and Pesaran, 2015, for a similar argument). The result (E.193) holds for the inverses due to the arguments in Andrews (1987). Moreover, $N_T\frac{1}{NT}\frac{\partial^2 \ell^*(\theta_2^0)}{\partial\theta_2(\partial\theta_2)'}N_T$ converges to a positive definite matrix by Lemma E.2. Combining the results,

$$\sqrt{NT}N_T^{-1}(\hat{\theta}_2 - \theta_2^0) \to_d N\left(0_{2\times 1}, \lim_{N,T\to\infty}\begin{bmatrix} \sigma_0^4(\kappa_0 - 1) & 0 \\ 0 & 1/s_0^2 \end{bmatrix}\right)$$

as $N, T \to \infty$. ■

Proof of Corollary 1.

We need to prove that Lemma E.2 (with obvious changes to the normalization) holds under the conditions of Corollary 1. The steps are the same as in Proof of Lemma E.2. The details are omitted but can be obtained upon request from the corresponding author. ■

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